IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XVIII, N. 5 *Serie decima* 1° Dicembre 1960

A Measurement of the Decay of ^{181}W.

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(ricevuto il 6 Maggio 1960)

Summary. — The half-life of the 59.2 keV X-radiation following the decay by orbital electron capture of ^{181}W has been measured to be (126.2±3.2) days. The measurement was made using a source electromagnetically enriched in ^{180}W before irradiation. In order to insure that the possible low-energy radiation from ^{185}W was not influencing the measurements, a calculation was made to show that this contamination was not a problem. An additional measurement on a natural tungsten source confirmed the results of the calculation.

1. – Introduction.

A reactor-produced radioactive tungsten source a few weeks after irradiation consists primarily of ^{181}W which decays by orbital electron capture with a half-life of 140 days ([1]), and ^{185}W which decays by β emission with a half-life of 74.5 days ([2-3]). These isotopes have been the subject of considerable controversy as is discussed in references ([2]) and ([3]) as well as in a number of other articles. It has become relatively clear however that the detectable

([1]) P. DEBRUNER, E. HEER, W. KÜNDIG and R. RÜETSCHI: *Helv. Phys. Acta*, **29**, 235 (1956).
([2]) B. P. SINGH and H. S. HANS: *Nuovo Cimento*, **14**, 108 (1959).
([3]) A. BISI and L. ZAPPA: *Nuovo Cimento*, **10**, 90 (1958).

radiation accompanying ^{181}W decay is the 59.2 keV (effective energy) X-radiation of ^{181}Ta following orbital electron capture. On the other hand ^{185}W decays by β emission to the ground state of ^{185}Re for the most part with some possibility that an excited state exists at 125 keV and possibly higher levels. In observing the decay of a reactor-produced natural tungsten source by detection of the 59.2 keV X-rays ([4]), we had previously observed half-lives that suggested a mixture of the 140 day and 74 day constituents. This tended to confirm the presence of radiation from ^{185}W at around 59 keV following internal conversion of a higher energy γ-ray or else as a primary γ-ray. Another possibility was that the half-life of ^{181}W was in reality less than the 140 day value reported in most of the previous literature on the subject. To attempt to resolve the discrepancy, a source was obtained which had been enriched in ^{180}W prior to bombardment in the ORNL reactor. The decay in intensity of the 59.2 keV radiation was then followed using a scintillation spectrometer after a suitable period had passed to allow for the decay in intensity of any ^{185}W radiation which might interfere with the measurement. It is this measurement which will be reported here.

2. – Experimental method.

The source used in this measurement was obtained from the Oak Ridge National Laboratory. The isotopic analysis of the enriched tungsten indicated the following atomic percentage of the constituents: ^{180}W, 6.95 percent; ^{182}W 42.16 percent; ^{183}W, 14.15 percent; ^{184}W, 22.22 percent and ^{186}W, 14.52 percent. The spectrographic analysis detected the following impurities: Cu, Fe Mg, Na, Si, Ti with relative abundances ranging from .01 to .08 percent. The source was irradiated in the period from Nov. 8, 1955 to Nov, 29, 1955.

The source material was encapsulated as dry WO_3 in a thin-walled (< 0.1 cm thick) glass capsule. The γ-ray spectrum was investigated using a 4 in. $\times$ 4 in. NaI crystal spectrometer (Fig. 1) and a 100-channel pulse-height analyzer. The crystal covering consisted of .010 in. Al plus a thin layer of MgO. The source was placed at a distance of $9\frac{1}{2}$ in. from the front surface of the crystal in a geometry that was accurately reproducible and the radiation was collimated to a diameter of $1\frac{1}{2}$ in. at the crystal by the opening in the shield surrounding the crystal-photomultiplier unit. Early measurements of the source spectrum showed γ-ray peaks at 59 keV, 134 keV, 480 keV, 550 keV, 618 keV, 690 keV and 775 keV, and the last six peaks were identified as belonging to the short-lived ^{187}W. The experimental γ-ray spectrum obtained at the beginning of the half-life determination reported here is shown in Fig. 2 and

([4]) W. E. KREGER, L. D. MCISAAC, J. L. MACKIN and J. R. LAI: *Phys. Rev.*, **100**, 955 (1955).

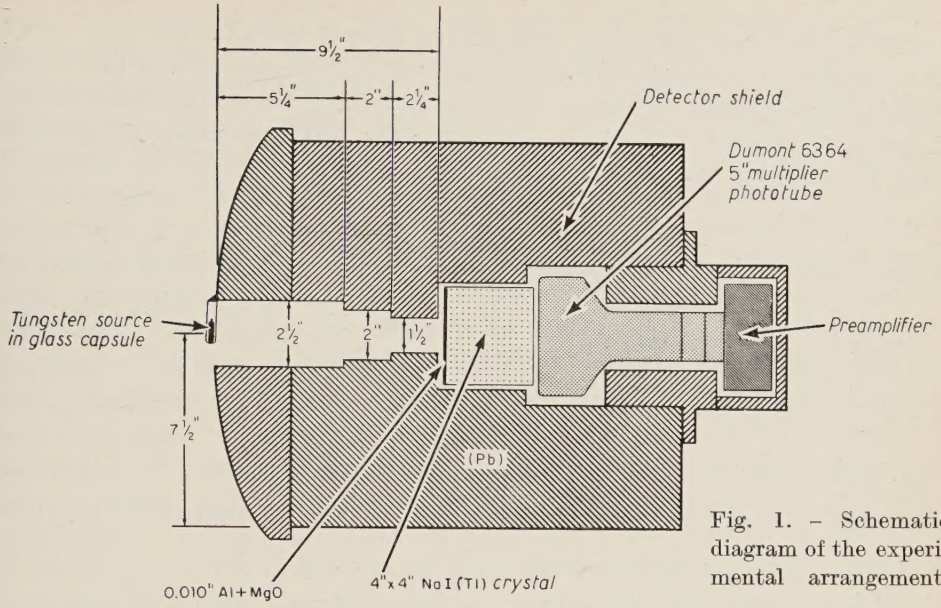

Fig. 1. – Schematic diagram of the experimental arrangement.

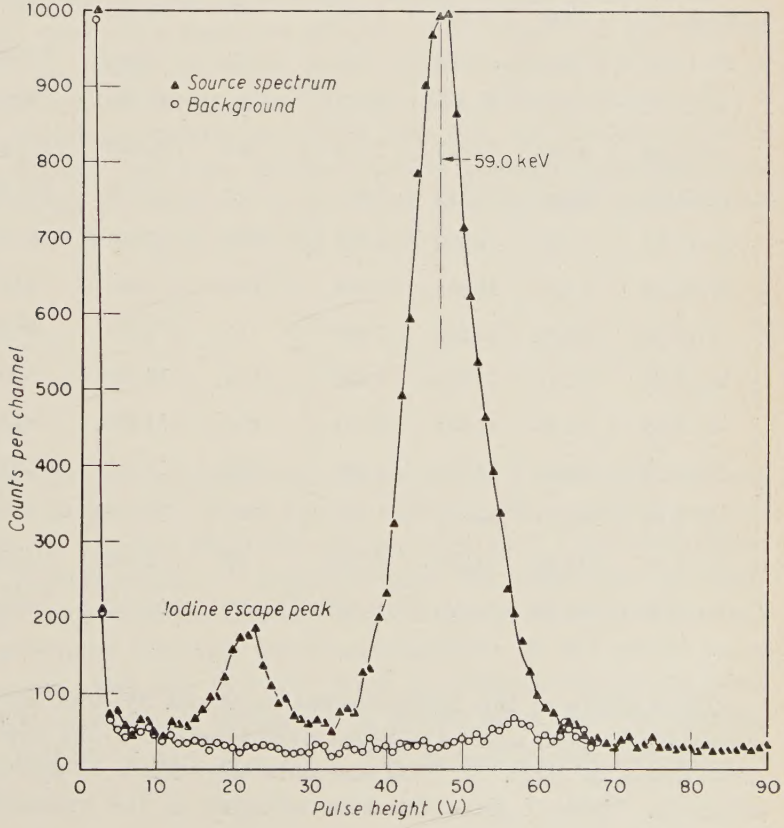

Fig. 2. – Pulse-height spectrum of the tungsten source as obtained with a 4 in. × 4 in. NaI crystal scintillation spectrometer. Counts were integrated between a pulse height of 11.8 V and 66.7 V.

consists of the total absorption peak due to characteristic X-rays of Ta which have an effective energy of 59.2 keV along with the associated iodine escape peak. The calibration of the spectrometer provided an energy value for the X-ray peak of (59.0 ± 0.5) keV, confirming the fact that these X-rays came from Ta following the orbital electron capture in ^{181}W.

TABLE I. – *Experimental data.*

Date	Time (days)	Total counts in peaks	Background	Continuum correction	Net counts	Exptl. peak count rate (min^{-1})	Best fit count rate (min^{-1})
21- 9-59	0	13 530	1 218	159	12 153	1 364.6	1 360.1
25- 9-59	3.98	14 216	1 292	88	12 836	1 341.7	1 330.6
2-10-59	11.04	13 503	1 231	88	12 184	1 279.0	1 280.1
9-10-59	17.96	14 088	1812	102	12 174	1 204.7	1 232.4
17-10-59	25.96	14 725	1 500	85	13 140	1 156.7	1 179.3
23-10-59	32.10	15 630	1 864	22	13 744	1 173.3	1 140.2
30-10-59	38.98	15 207	1 476	120	13 611	1 163.0	1 098.0
6-11-59	45.94	14 437	1 715	280	12 442	1 022.3	1 056.8
11-11-59	50.98	14 218	1 640	92	12 486	1 015.9	1 027.9
20-11-59	60.03	14 896	2 092	313	12 491	957.9	978.1
27-11-59	67.13	14 335	1 784	190	12 361	944.2	940.8
4-12-59	73.91	14 230	1 640	15	12 575	908.3	906.4
14-12-59	84.11	15 234	2 650	118	12 466	860.3	857.0
18-12-59	87.95	14 834	2 701	204	11 929	842.4	839.1
28-12-59	98.10	14 198	1 968	324	11 906	781.7	793.7
4- 1-60	105.14	14 061	1 887	168	12 006	760.9	763.5
11- 1-60	112.11	14 317	2 041	93	12 183	741.8	734.8
19- 1-60	120.95	14 257	2 211	84	11 962	704.8	715.6

The half-life of this radiation was measured by determining the time decrease in the area under the peaks over a period of 121 days. Measurements were begun on Sept. 21, 1959 and data were taken at approximately weekly intervals. Table I gives the data obtained in the measurements described

above. The counts in the peaks were added for the energy interval from 16.1 keV to 83.4 keV, as established by the energy calibration provided in each run by the position of the 59 keV peak and the iodine escape peak. These counts are shown in column 3 of Table I. The energy limits where chosen to include all the counts in the 59 keV total absorption peak and associated iodine escape peak. A background run was made for each source run and the background counts (column 4 of Table I) in the same energy interval were subtracted from the source run. In addition to the two predominant peaks, there appeared to be a continuous distribution of pulses at energies above the energy of the X-ray peak which were attributed to bremsstrahlung from the β-decay of ^{185}W. In order that these not influence the half-life measurement, the difference between the source run count and background counts was determined in the energy interval from 85 keV to 112 keV and used to determine the average counts per channel which might be under the 59 keV peak, assuming a flat distribution of bremsstrahlung pulses over the energy interval from 16.1 keV to 112 keV. This number was multiplied by the number of channels involved in the 59 keV peak count and subtracted as a potential contamination. The contribution from this source (column 5 of Table I) amounted to about 0.8 percent of the total counts in the X-ray peak.

The spectrum runs were made to provide 1000 counts in the channel representing 59 keV, which provided about 14000 counts (column 3 of Table I) in the integrated energy interval used. The time was measured in a true time device which determines the actual time that the analyzer is available to accept pulses. The time measurement is accurate to better than 0.1 percent, providing an accuracy in the count rate for each measurement of ± 0.9 percent due mostly to counting statistics.

3. – Results.

The results of these measurements are shown in Fig. 3 where the counting rate for each measurement is plotted against time. A least squares determination of the half-life based on these data gives a value of $T_{\frac{1}{2}} = (126 \pm 3.2)$ days.

Since several experimenters have suggested that ^{185}W might provide radiation in the energy range used for this half-life determination, a calculation was undertaken to determine just how much contamination in the energy range around 59 keV could exist due to ^{185}W. Such a contamination would make the half-life of the combination appear to be shorter than the ^{181}W half-life, since ^{185}W has a half-life of only 74.5 days. The calculation was based on the following factors:

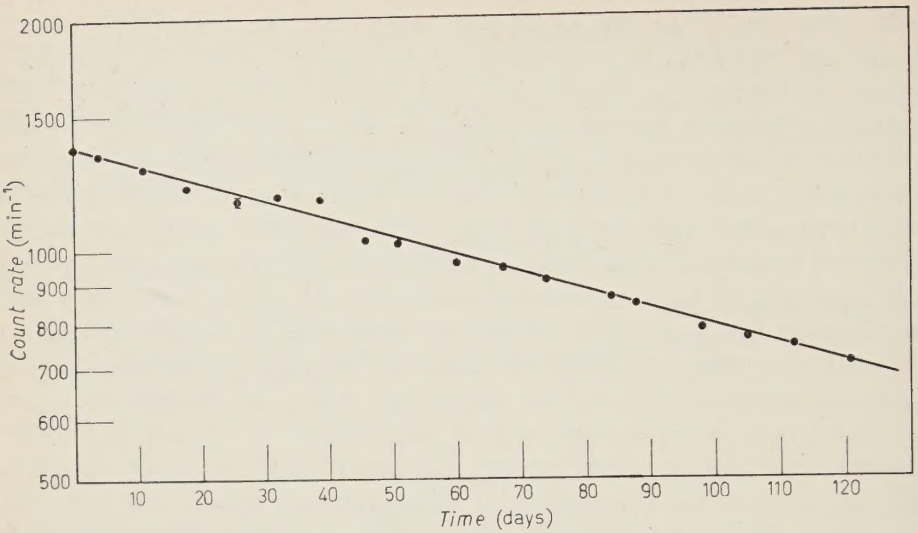

Fig. 3. – Decay curve of the radiation in the interval from 16.1 keV to 83.4 keV from ^{181}W. Counting error is indicated on the point at 26 days.

1) The thermal neutron capture cross-section for ^{180}W has been reported as 30 barns $^{+400\ \text{percent}}_{-100\ \text{percent}}$ ([5]) and as 2 barns ([6]) with a possible error of a factor of ten. The thermal neutron capture cross-section for ^{184}W is reported as 2.12 barns ([7]) and as 1.97 barns ([5]). Assuming that pile-produced radioactive tungsten results primarily from thermal neutron capture, and choosing the lowest value of the ^{180}W cross-section from the point of view of an unfavorable ^{185}W/^{181}W ratio, we get a possible contamination ratio at production of $^{185}\text{W}/^{181}\text{W} = 22.22 \times 2.04\ \text{barns}/6.95 \times 2\ \text{barns} = 3.26$.

2) The total decay time before the start of the experiment was 1326 days, reducing the contamination ratio to $^{185}\text{W}/^{181}\text{W} = 3.26 \cdot 5.07 \cdot 10^{-3} = 1.653 \cdot 10^{-2}$ This was based on a $T_{\frac{1}{2}}(^{181}\text{W}) = 126.2$ days and $T_{\frac{1}{2}}(^{185}\text{W}) = 74.5$ days.

3) The upper limit of the intensity of possible γ-rays from ^{185}W in the energy interval from 33 keV to 88 keV has been reported to be $3.5 \cdot 10^{-3}$ per disintegration ([8]). It has also been measured at 60.1 keV to be $1.54 \cdot 10^{-3}$ γ-rays per β-particle ([3]). Using the former value for ^{185}W and assuming one

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([5]) H. POMERANCE: *Phys. Rev.*, **88**, 412 (1952).
([6]) M. LINDNER: *Phys. Rev.*, **84**, 240 (1951).
([7]) L. SEREN, H. N. FRIEDLANDER and S. H. TURKEL: *Phys. Rev.*, **72**, 888 (1947).
([8]) V. S. DUBEY, C. E. MANDEVILLE, A. MUKERJI and V. R. POTNIS: *Phys. Rev.*, **106**, 785 (1957).

X-ray per disintegration in ^{181}W, we calculate a ratio of decay rate of 59 keV radiation from ^{185}W to decay rate of the same energy radiation from ^{181}W at the beginning of this experiment to be $^{185}R/^{181}R = 4.7 \cdot 10^{-5} \cdot 1.653 \cdot 10^{-2}/5.5 \cdot 10^{-3} = 1.41 \cdot 10^{-4}$.

This result would lead to the conclusion that the measured half-life value is not affected by ^{185}W contamination.

In order to confirm some of the questionable constants used in the above calculation, a natural tungsten source was obtained and a measurement made of its half-life in exactly the same kind of experiment as is reported here. The half-life value obtained was 115.6 days. This value could be obtained with a source having decay rates at the time of the run in the ratio $^{185}R/^{181}R = .19$. A calculation using the same constants as in the above calculation with the exception that natural instead of enriched tungsten was irradiated, and a shorter period had occurred before the half-life measurement was made, gave a decay rate ratio $^{185}R/^{181}R = .34$. This calculation predicts an even higher tungsten 185 to tungsten 181 decay rate ratio than is observed experimentally. Therefore the calculation which was used above and predicted a tungsten 185 to tungsten 181 decay rate ratio of $1.41 \cdot 10^{-4}$ for the source used in the measurement must be conservative, suggesting an even lower actual ratio. It would therefore seem clear that ^{185}W contamination cannot have influenced the half-life measurement described in this paper.

One additional check has been made on this point. It was assumed that the source did consist of 140 day ^{181}W and 74.5 day ^{185}W and a least squares best fit was made between the experimental data and a mixture of these two activities. The adjustable parameters used to obtain the best fit were the amount of ^{181}W and ^{185}W present at the beginning of the experiment. This best fit calculation gave a value of 0.183 for the ratio $^{185}R/^{181}R$ at the beginning of the experiment, which is in considerable disagreement with $1.41 \cdot 10^{-4}$. The standard deviation of the mixture count rate values from the experimental values was 23.3, while the standard deviation obtained using a 126 day straight line and the experimental values was 23.4. This would indicate that the single half-life straight line provides the same fit to the experimental data as a possible mixture of the 140 day and 74.5 day tungsten isotopes.

We therefore conclude that the decay of ^{181}W occurs with a half-life value of $T_{\frac{1}{2}} = (126 \pm 3.2)$ days.

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The authors would like to thank T. H. Jones for his assistance in making the measurements reported here.

RIASSUNTO (*)

Si è misurata in (126.2±3.2) giorni la vita media della radiazione X di 59.2 keV del ^{181}W che segue al decadimento per cattura orbitale dell'elettrone. La misura fu eseguita usando uno sorgente arricchita elettromagneticamente di ^{180}W prima della irradiazione. Allo scopo di assicurarsi che una possibile radiazione di bassa energia del ^{185}W non influenzasse le misurazioni, si è eseguito un calcolo per dimostrare che tale contaminazione non è un problema. Una misurazione addizionale su una sorgente di tungsteno naturale confermò i risultati del calcolo.

(*) *Traduzione a cura della Redazione.*

Calculation of the Vapour Pressure Ratio of Ne, A, Kr, and Xe Isotopes in the Solid State.

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(ricevuto il 2 Luglio 1960)

Summary. — In order to discuss the validity of the anharmonic Einstein model developed by Henkel, the vapour pressure ratio of neon isotopes is recalculated, following Johns's method, by making use of a Lennard-Jones (6-12) potential and of different couples of parameters ε and σ. Results are compared with the recent experimental data by Roth and Bigeleisen. The dependence of $\ln p_m/p_M$ from temperature is found to be scarcely sensitive to the choice of molecular parameters and is in good agreement with experimental results. As to the absolute values of $\ln p_m/p_M$ at any given temperature one does not find as good an agreement as claimed by Johns. It is shown how the uncertainty in the determination of molecular parameters cannot explain the whole disagreement between experiments and theory. Calculations for argon are carried on in the same scheme. Results are lower than preliminary experimental results by Boato and Scoles. A brief discussion is given to suggest why difference in vapour pressure between two separated isotopes could be slightly different from the same quantity deduced by measuring the fractionation factor of an isotopic mixture. A possible influence of the isotopic composition of an isotopic solid mixture on the fractionation factor is also pointed out. Finally the $\ln p_m/p_M$ for krypton and xenon is calculated by means of the quantum theorem of corresponding states.

1. – Introduction.

In the last years there has been considerable interest in an extension, deve'oped by Henkel [1], of the Einstein lattice model. This method is ap-

[1] J. H. Henkel: *Journ. Chem. Phys.*, **23**, 681 (1955).

plicable to solids when the interaction forces are supposed to be represented by a Lennard-Jones $(m - n)$ potential, and takes into account, in the scheme of perturbation method, the effect of the anharmonic terms in the cell potential energy. It has been used by several authors [1-5] to calculate equilibrium thermodynamic properties of simple solids.

In order to discuss its limits of validity, we found particularly interesting a detailed analysis of the dependence on temperature of the vapour pressure ratio of heavy noble gases in the solid phase.

This ratio has already been calculated for neon by Johns by using a Lennard-Jones (6–14) potential. Calculations for neon were repeated with a (6–12) potential, given by $\varphi(r) = 4\varepsilon((\sigma^{12}/r^{12}) - \sigma^6/r^6)$, and with various couples of the parameters ε and σ; the results were compared with the recent experimental data by ROTH [6,7]. Calculations have also been carried out for argon, to be compared with the experiments in progress by BOATO and co-workers at Genoa University.

Our line of approuch to the problem is substantially the same as Johns's: we express the free energy through the partition function and find the equilibrium value of the lattice constant a corresponding to the minimum value of f. The difference $f_m - f_M$ is evaluated from the free energies of the two isotopes by the assumption that the lattice constant has the same value for the two isotopes.

2. – The scheme of the calculations.

By assuming that the vapour behaves as a perfect gas, the vapour pressure ratio of two isotopes is given by the expression [4]

$$\ln \frac{p_m}{p_M} = \frac{3}{2} \ln \frac{m}{M} + \frac{f_m - f_M}{kT}, \tag{1}$$

where m and M are the masses of the two isotopes, f_m and f_M their Helmholtz free energies in the solid phase, k the Boltzman constant and T the absolute temperature. Our task is hence to evaluate the free energy of the two isotopes at the various temperatures of interest. It is necessary to point out that the free energy must be calculated at a value a_e of the lattice constant, relative to the equilibrium state $(\mathrm{d}f/\mathrm{d}a)_{a=a_e} = 0$. In fact, due to inaccuracy

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(2) C. DOMB and I. J. ZUCKER: *Nature*, **178**, 484 (1956).
(3) I. J. ZUCKER: *Journ. Chem. Phys.*, **25**, 915 (1956).
(4) J. F. JOHNS: *Phil. Mag.*, **3**, 229 (1958).
(5) E. R. DOBBS and G. O. JONES: *Rep. Progr. Phys.*, **20**, 516 (1957).
(6) E. G. ROTH: private communication.
(7) E. G. ROTH and J. BIGELEISEN: *Journ. Chem. Phys.*, **32**, 612 (1960).

of experimental data and inadequacy of theory, one does not always have an accurate correspondence between the value of a_e theoretically calculated by minimizing the free energy and the lattice constant obtained by extrapolation of X-rays diffraction measurement ([8]) or by neutron diffraction ([9]) (Fig. 1-*a*). In Fig. 1-*b* it is shown how, for argon at $T = 80$ °K, an error of $\delta a = 0.01$ Å the lattice constant can affect considerably the value of $f_m - f_M$.

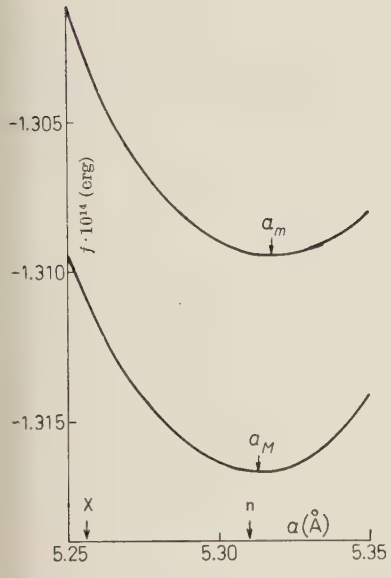

Fig. 1*a*. – Free energy of ^{36}A and ^{40}A, at $T = 4.2$ °K as a function of the lattice constant. The upper curve corresponds to the lighter isotope; the equilibrium lattice constant a_m is slightly higher than a_M (heavier isotope, lower curve). The value of the lattice constant is slightly higher for the higher for the lighter isotope to which the upper curve is referred. The lattice constants obtained by neutron diffraction (n) and by extrapolation of X-rays diffraction measurements (X) are also shown.

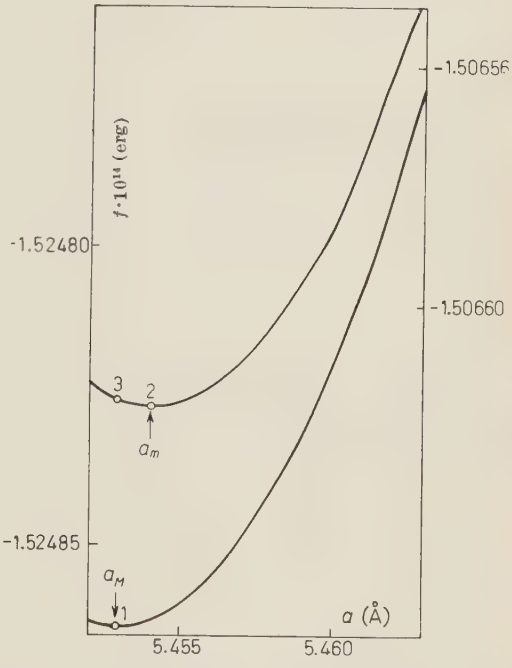

Fig. 1*b*. – Free energy of ^{36}A and ^{40}A at $T = 80$ °K as a function of the lattice constant. The right-hand scale is relative to the upper curve (^{36}A).

To calculate f we assume, as usual, that interatomic forces can be represented by a Lennard-Jones (6—12) potential. Each atom in the crystal is supposed to vibrate indipendently of its neighbours, fixed in their equilibrium position. Every correlation between the motion of two atoms is therefore neglected. Effects due to non-additivity of Van der Waals forces

([8]) E. R. Dobbs, B. G. Figgins, G. O. Jones, D. C. Piercey and D. P. Riley: *Nature*, **178**, 483 (1956).

([9]) D. G. Henshaw: *Phys. Rev.*, **111**, 1470 (1958).

are also neglected. According to HENKEL, to obtain the anharmonic potential acting on every atom we add the contributions due to all the atoms of the crystal, and develop in series the resulting potential function up to fourth order terms in x, y and z, the components of the displacements from lattice point; mixed terms of the fourth order are simmetrized and the anharmonic potential energy in the lattice cell reads:

$$P(x, y, z) = \frac{P_0}{2} + P_2(x^2 + y^2 + z^2) + P_4(x^4 + y^4 + z^4) \; ; \tag{2}$$

The coefficients P_0, P_2 and P_4 are expressed through lattice sums [1] (*).

When the isotopic effect is studied for the two separated isotopes, the difference $\Delta f = f_m - f_M$ concerns the values of free energies in the two different equilibrium configurations, it is namely

$$\Delta f = f_m(a_m) - f_M(a_M) \; . \tag{3}$$

We know, however, that the difference in the lattice spacing between the two isotopes is very small (at $T = 0\,^\circ\mathrm{K}$, $\Delta a/a = (a_m - a_M)/a_m \simeq 5 \cdot 10^{-4}$ for argon (see Fig. 1-*a*), and $\Delta a/a \simeq 2 \cdot 10^{-3}$ for neon), and we can write, without appreciable error

$$\Delta f = f(a_m, m) - f(a_m + \Delta a, m + \Delta m) \simeq f(a_m, m) - f(a_m, m + \Delta m) - \left[\frac{\partial f(a, m)}{\partial a}\right]_{a_m, m} \Delta a = f(a_m, m) - f(a_m, m + \Delta m) \; , \tag{4}$$

since $\partial f/\partial a = 0$ at the equilibrium. To evaluate Δf it is hence sufficient to

(*) In order to apply the perturbation method to the Schrödinger equation for the anharmonic isotropic oscillator we must ensure that the anharmonic energy $P_4(x^4 + y^4 + z^4)$ is considerably smaller than the harmonic energy. This condition yields, in our case

$$P_4 \ll \frac{(32 m P_2^3)^{\frac{1}{2}}}{3\hbar} \cdot \frac{n + \frac{1}{2}}{n^2 + n + \frac{1}{2}} \equiv G(T) \; .$$

Here n is evaluated from the expression

$$n = n_x = n_y = n_z = \left[\exp\left[\frac{h\nu_0}{kT}\right] - 1\right]^{-1}, \qquad \nu_0 = \frac{1}{2\pi}\sqrt{\frac{2P_2}{m}} \; ,$$

obtained by associating an average number n of quanta to every polarized wave. We found that the method is applicable to solid argon and neon: at the triple point the ratio $P_4/G(T_{tr})$ is 0.06, in the case of argon, and 0.14 in the case of neon; this ratio is smaller at lower temperatures.

minimize $f(a)$ for one isotope only. This holds however, when quantum effects are small, as in the case of neon and argon.

3. – Results of calculations for neon and discussion.

The vapour pressure ratio for neon was calculated in the same scheme by JONES (4) two years ago by using a (6–14) Lennard-Jones potential proposed by ZUCKER (3) from an analysis of the data of neon isotherms given by Stewart. Johns's calculations were compared with experimental data from KEESOM and HAANTJIES (10). Some more experimental data are now available: Stewart isotherms have been extended (11) in a wider range of pressure, and from these data ZUCKER (12) himself found thata better agreement with theory is obtained by using a (6–12) potential; the lattice constant of neon has been measured by HENSHAW (9) by neutron diffraction; finally more recently the $\ln p_m/p_M$ of solid neon isotopes has been measured (6,7) from 16 °K to triple point. Calculations have been repeated with several couples of parameters. In Table I, *a*) is the new potential from Zucker; *b*) has been choosen to give, in Henkel's scheme, the Henshaw value for the lattice constant and the calculated value of the sublimation energy at 0 °K (*); *c*) has been deduced from the quantum

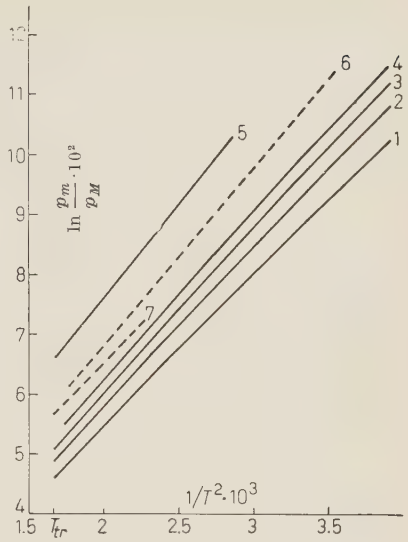

Fig. 2. – The logarithm of the vapour pressure ratio of neon isotopes ^{20}Ne and ^{22}Ne *vs* T^{-2}. Curves 1), 2), 3), have been obtained by using a (6-12) potential with the following couples of parameters. 1) $\varepsilon = 4.97 \cdot 10^{-5}$ erg, $\sigma = 2.831$ Å; 2) $\varepsilon = 5.00 \cdot 10^{-15}$ erg, $\sigma = 2.756$ Å; 3) $\varepsilon = 5.21 \cdot 10^{-15}$ erg, $\sigma = 2.756$ Å. Curve 4) is the theoretical curve by Johns relative to a (6-14) potential, corrected using for the Boltzmann constant the value $K = 1.38047 \cdot 10^{-16}$ erg °K^{-1}. Curve 5) has been obtained neglecting anharmonicity. Curves 6) and 7) are experimental curves from ROTH (6) and KEESOM and HAANTJIES (10) respectively.

(10) W. H. KEESOM and J. HAANTJIES: *Physica*, **2**, 986 (1935).

(11) J. W. STEWART: *Phys. Rev.*, **97**, 578 (1955); *Journ. Phys. Chem. Sol.*, **1**, 146 (1956).

(12) I. J. ZUCKER: private communication (1958).

(*) For $\varepsilon = 5.00 \cdot 10^{-15}$ erg and $\sigma = 2.756$ Å one has, at $T = 4.2$ °K, $a_e = 4.433$ Å, in agreement with the experimental value of Henshaw $a_e = 4.429$ Å; the theoretical value of the sublimation energy is $L_0 = 3.075 \cdot 10^{-14}$ erg atom^{-1} and the value obtained from experimental data on latent heat of sublimation at triple point and specific heat as a function of temperature is $3.08 \cdot 10^{-14}$ erg atom^{-1}.

TABLE I

T (°K)	*a* $\varepsilon = 4.97 \cdot 10^{-15}$ erg $\sigma = 2.8312$ Å		*b* $\varepsilon = 5.00 \cdot 10^{-15}$ erg $\sigma = 2.756$ Å		*c* $\varepsilon = 5.21 \cdot 10^{-15}$ $\sigma = 2.690$ Å	
	$\Delta f \cdot 10^{16}$ (erg)	$\ln p_{20}/p_{22}$	$\Delta f \cdot 10^{16}$ (erg)	$\ln p_{20}/p_{22}$	$\Delta f \cdot 10^{16}$ (erg)	ln
0	5.052	—	5.168	—	—	
12	5.176	0.1695	5.319	0.1781	5.509	0
16	5.432	0.1030	5.560	0.1088	—	
20	5.818	0.0678	5.931	0.0719	6.092	0
22	6.052	0.0563	6.158	0.0598	—	
24	6.308	0.0474	6.408	0.0504	6.512	0
24.5	6.375	0.0455	6.484	0.0487	—	

a) parameters from Zucker:
b) parameters chosen to fit lattice distance and sublimation energy;
c) parameters deduced from the quantum theorem of corresponding states;
d-e-f-g) parameters arbitrarily chosen.

theorem of corresponding states ([13]); *d*), *e*), *f*), *g*), have been arbitrarily chosen in order to investigate how the results depend on the choice of ε and σ.

Results are shown also in Fig. 2. The absolute value of $\ln (p_m/p_M)$, calculated with any couples of parameters is, at any temperature, lower than the

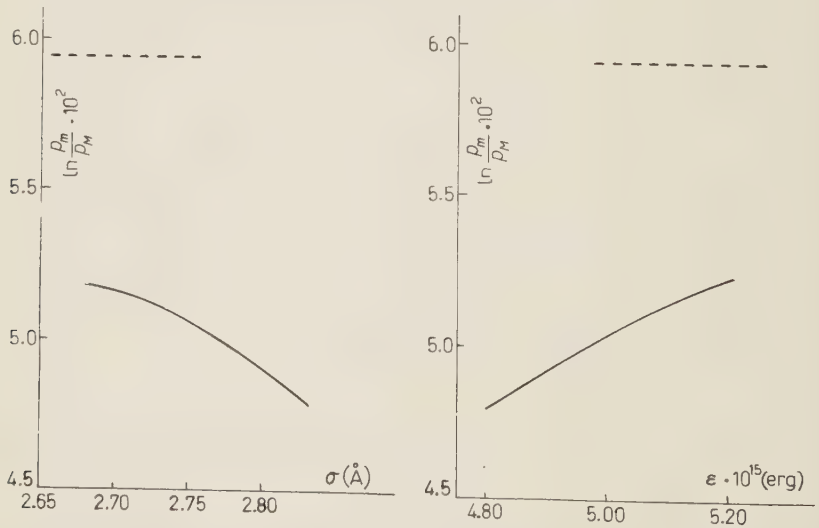

Fig. 3. – *a*) $\ln p_{20}/p_{22}$ as a function of σ (ε fixed); $T = 24$ °K and $\varepsilon = 5.00 \cdot 10^{-15}$ erg; *b*) $\ln p_{20}/p_{22}$ as a function of ε (σ fixed); $T = 24$ °K and $\sigma = 2.756$ Å. The dotted line indicates the experimental value from ROTH.

([13]) G. BOATO and G. CASANOVA: to be published.

d $80 \cdot 10^{-15}$ erg 756 Å		*e* $\varepsilon = 5.00 \cdot 10^{-15}$ erg $\sigma = 2.720$ Å		*f* $\varepsilon = 5.00 \cdot 10^{-15}$ erg $\sigma = 2.8312$ Å		*g* $\varepsilon = 5.21 \cdot 10^{-15}$ erg $\sigma = 2.756$ Å	
rg)	$\ln p_{20}/p_{22}$	$\Delta f \cdot 10^{16}$(erg)	$\ln p_{20}/p_{22}$	$\Delta f \cdot 10^{16}$(erg)	$\ln p_{20}/p_{22}$	$\Delta f \cdot 10^{16}$(erg)	$\ln p_{20}/p_{22}$
	—	—	—	5.055	—	—	—
	0.1693	5.341	0.1794	5.185	0.1622	5.402	0.1831
	0.1031	5.586	0.1099	5.444	0.1035	5.640	0.1124
	0.0681	—	—	5.832	0.0683	6.005	0.0745
	—	—	—	—	—	—	—
	0.0479	6.437	0.0513	6.324	0.0479	6.473	0.0524
	—	—	—	—	—	—	—

experimental value, whereas the dependence on temperature is satisfactory. Fig. 3-*a* and 3-*b* show the dependence of $\ln p_m/p_M$ on σ (ε fixed) and on ε (σ fixed) at $T = 24$°K. The calculated value of $\ln p_m/p_M$ increases with increasing ε and decreasing σ. The parameters *e*) are the higher value of ε and the lower values of σ proposed in literature ([13]). We conclude that the uncertainty in the determination of molecular parameters cannot explain the whole disagreement between experiments and theory.

A comparison with the results obtained by JOHNS with a (6–14) potential suggests that an exponent in the repulsive potential still greater than 14 should give a better fitting with experimental data. The need of a high exponent in the repulsive potential is indicated, for argon, in a recent paper by BROWN and ROWLINSON ([14]). A more substantial reason for the disagreement might be due to the neglecting of the coupling of vibrations of molecules: by using a Debye's model, for instance, one finds that the vibrational energy at 0 °K is greater than in the Einstein model; it follows then that also the difference between vibrational levels of the two isotopes, to which the isotopic effect is related, is greater.

On the other hand, it is a hard job to keep into account both coupling of the motion of the molecules and anharmonicity, and one does not know which of these two effects is greater. It is worth noting that in our calculations the $\ln p_m/p_M$ obtained by neglecting anharmonicity (Fig. 2) is 10% higher and is also higher than experimental values.

([14]) W. B. BROWN and J. S. ROWLINSON: *Molecular Phys.*, **3**, 35 (1960).

4. – Results of calculations for argon.

Table II and Fig. 4 show the results of the calculations for argon, carried on with the values of the parameters given by DOMB and ZUCKER [2]:

TABLE II. – *Argon results.*

T (°K)	*a* $\varepsilon = 1.69 \cdot 10^{-14}$ erg, $\sigma = 3.402$ Å		*b* $\varepsilon = 1.65 \cdot 10^{-14}$ erg, $\sigma = 3.41$ Å		*c* $\varepsilon = 1.71 \cdot 10^{-14}$ erg, $\sigma = 3.36$ Å	
	$\Delta f \cdot 10^{15}$ (erg)	$\ln p_{36}/p_{40}$	$\Delta f \cdot 10^{15}$ (erg)	$\ln p_{36}/p_{40}$	$\Delta f \cdot 10^{15}$ (erg)	$\ln p_{36}/p_{40}$
0	—	—	—	—	—	—
20	0.7768	0.1232	—	—	—	—
40	1.0530	0.0326	—	—	—	—
60	1.4228	0.0137	—	—	1.4275	0.0143
75	1.7224	0.0083	—	—	1.7261	0.0087
80	1.8246	0.0072	1.8218	0.0069	1.8280	0.0075
83.78	1.9025	0.0065	—	—	1.9040	0.0067

a) Domb-Zucker.
b) De Boer.
c) arbitrarily chosen.

$\varepsilon = 1.69 \cdot 10^{-14}$ erg, $\sigma = 3.402$ Å. At present the comparison with experiments is more difficult, as accurate experimental data are not yet available. The broken line in Fig. 4, has been constructed by shifting the theoretical curve to fit the datum at $T = T_{tr}$ deduced from the quantum theorem of corresponding states; on this line lie the preliminary experimental results obtained by BOATO and SCOLES [15] at Genoa University. One can see that the theoretical effect is $\simeq 10\%$ too low. This discrepancy is confirmed by considering the experimental results of the liquid at triple

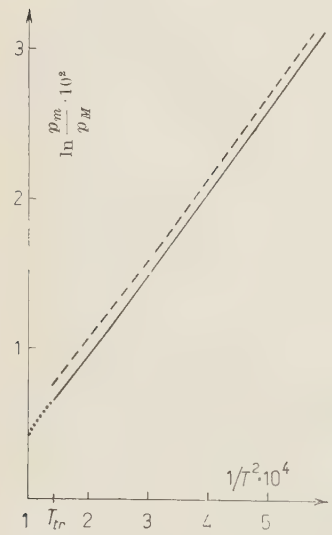

Fig. 4. – $\ln p_m/p_M$ versus T^{-2} for the argon isotopes ^{36}A and ^{40}A. Solid line: theoretical values; broken line: values shifted to fit the datum at $T = T_{tr}$ obtained from the principle of corresponding states; dotted line: experimental results for liquid mixtures [16].

[15] G. BOATO and G. SCOLES: private communication.

point ([16]) (Fig. 4). It appears that both for argon and neon theoretical results are lower than experimental data.

Calculations have been repeated by using ε and σ given by DE BOER ([17]) and other parameters arbitrarily chosen; the isotopic effect is still too low. In order to obtain a fitting with experimental data one should change considerably ε and σ: here again we are led to the conclusion that the theoretical scheme cannot give a very accurate description of the isotopic vapour pressure ratio.

5. – Comparison between theoretical scheme and experimental methods.

In order to compare the results of the calculations with the experimental data one should distinguish, in an accurate analysis, between values of the vapour pressure ratio obtained by direct measurements ([6,10]), or evaluated from the fractionation factor of a mixture of the two isotopes ([16]). The theoretical scheme here adopted is directly related to the first type of measurements, since formula (1) is based on the calculation of the free energy of the two separated isotopes.

According to PRIGOGINE *et al.* ([18]), if v_M is the atomic volume of the heavy isotopes, v_m the atomic volume of the light one and v the « average » atomic volume in the solid mixture (*), it is sufficient to evaluate the variation of free energy associated to the variation of volume of each isotope in the mixture. One should account for this variation assuming to exert a pressure on the heavy isotope to increase the volume from v_M to v, and a « negative » pressure on the light one:

$$(5) \qquad \begin{cases} \sigma f_m = f_m(v) - f_m(v_m), \\ \sigma f_M = f_M(v) - f_M(v_M). \end{cases}$$

Referring to Fig. 1-*b*, the difference between the minimum values assumed by f_m and f_M (points 2 and 1) is related to the vapour pressure ratio of pure isotopes, while the « vertical » difference (points 1, 3) corresponding to the average lattice constant leads to the evaluation of the isotopic effect in the mixture.

([16]) G. BOATO, G. SCOLES and M. E. VALLAURI: *Nuovo Cimento*, **14**, 735 (1959); G. BOATO, G. CASANOVA and M. E. VALLAURI: *Nuovo Cimento*, **16**, 505 (1960).

([17]) J. DE BOER: *Physica*, **14**, 139 (1948).

([18]) I. PRIGOGINE, R. BINGER and A. BELLEMANS: *Physica*, **20**, 633 (1954); I. PRIGOGINE: *The Molecular Theory of Solutions* (Amsterdam, 1957).

(*) In a mixture however the cells of the two isotopes are not exactly the same, since the lighter isotope should vibrate in a cell slightly larger than that of the heavier isotope.

In natural argon the heavy isotope ^{40}A is by far the more abundant; the average lattice constant will then be nearer to a_M and $\ln p_m/p_M$ in the mixture is slightly higher than $\ln p_m/p_M$ measured on pure isotopes. In the case of neon the situation is reversed, as ^{20}Ne is more abundant than ^{22}Ne.

The $\ln p_m/p_M$ calculated from the fractionation factor of the natural isotopic mixture is 1.5‰ higher at 80 °K for argon, and 6‰ lower at 24 °K for neon, than the same quantity for pure isotopes. It also appears that the fractionation factor of a solid mixture of neon isotopes at 24 °K with prevailing ^{22}Ne should lead to a value of $\ln p_m/p_M \simeq 1\%$ higher than the value relative to the natural mixture. This could be tested by experiment.

As to calculation on A and Ne the previous difference is obviously negligible, but could be relevant for substances with high quantum effect.

6. – The principle of corresponding states and the isotopic effect for krypton and xenon.

As to the extention of this kind of calculation to the heavy noble gases, one should point out that, since $\ln p_m/p_M$ is very small ($\simeq (1/100) \ln m/M$), formula (1), as well as any other similar expression obtained from statistical termodynamics, is less adequate, since one should compute with the greatest accuracy ($\sim 10^{-4}$) the difference between the free energies of the two isotopes in the solid state.

On the other hand the vapour pressure ratio of krypton and xenon can be obtained by means of the quantum theorem of corresponding states ([17]), when the vapour pressure ratio of argon is known. If the same law of the interaction among atoms in the solid holds for all noble gases, one has a set of reduced isotherms $p^* = p^*(\Lambda^*)$ for the reduced vapour pressure of all noble gases; as usual, the reduced vapour pressure is defined as $p^* = \sigma^3 p/\varepsilon$ and Λ^* is the so-called quantum parameter, or reduced De Broglie wave-length: $\Lambda^* = h/\sigma\sqrt{m\varepsilon}$. The isotherms $p^*(\Lambda^*)$ can be expanded in powers of Λ^{*2} ([19]),

$$p^* = p_{\text{cl}}^* + p_1^* \Lambda^{*2} + p_2^* \Lambda^{*4} + \dots , \tag{6}$$

and, for argon, krypton and xenon, powers higher than Λ^{*2} can be neglected One has then the following relation for the vapour pressure ratio:

$$\frac{1}{\Delta \Lambda^{*2}} \frac{\Delta p}{p} \simeq \frac{1}{\Delta \Lambda^{*2}} \ln \frac{p_m}{p_M} = p_1^* = \text{const} , \tag{7}$$

where $\Delta \Lambda^{*2}$ is taken from the Λ^*'s of the two isotopes considered.

([19]) J. DE BOER and R. J. LUNBECK: *Physica*, **14**, 520 (1958); R. J. LUNBECK: *Doctoral Dissertation* (Amsterdam, 1951).

The values of $\ln p_m/p_M$ for krypton and xenon have been obtained from (7), by equating $1/\Delta\Lambda^{*2} \ln p_m/p_M$ to the value of this functions for argon (broken line in Fig. 4), at the same reduced temperature. Λ^* and $T^* = kT/\varepsilon$ have been determined by means of the parameters ε and σ proposed by Boato and Casanova [13] by interpolation of various experimental data according to the quantum theorem of corresponding states:

$$\varepsilon(\mathrm{Kr}) = 2.28 \cdot 10^{-14} \text{ erg}, \qquad \sigma(\mathrm{Kr}) = 3.67 \text{ Å};$$
$$\varepsilon(\mathrm{Xe}) = 3.17 \cdot 10^{-14} \text{ erg}, \qquad \sigma(\mathrm{Xe}) = 2.95 \text{ Å}.$$

Results are plotted in Figs. 5 and 6.

It is worth noting that calculations carried out for neon show that the theoretical scheme followed in this paper, namely use of formula (1) and of Henkel's anharmonic approximation, seems to be consistent with the quantum

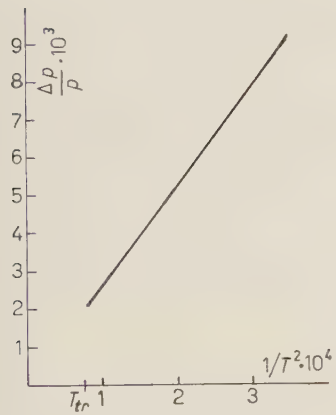

Fig. 5. – $\Delta p/p$ versus T^{-2} for the krypton isotopes ^{78}Kr and ^{86}Kr, deduced from the quantum theorem of corresponding states.

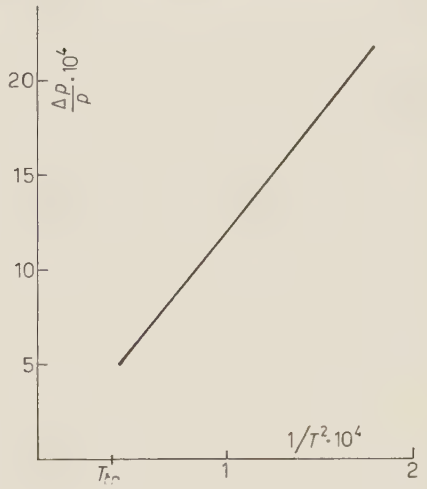

Fig. 6. – $\Delta p/p$ versus T^{-2} for xenon isotopes ^{128}Xe and ^{136}Xe, deduced from the quantum theorem of corresponding states.

principle of corresponding states; the parameters ε and σ of Table I column *a* and *c*, correspond, in fact, to the same value of Λ^*, and the two corresponding plots of $\ln (p_m/p_M)$ versus the reduced temperature are coincident.

* * *

The present work was carried out with the financial help of the « Comitato per la Fisica del Consiglio Nazionale delle Ricerche ». Part of the numerical calculations have been performed by the Electronic Computer of Genoa University.

The authors gratefully acknowledge Prof. G. Boato for many useful and extensive discussions, Dr. G. Osti of the I.N.F.N. of Milan, and Dr. L. Rebolia of the Computing group of Genoa for the help in the calculations. Two of us (G.C. and N.T.) would like to thank the « Consiglio Nazionale delle Ricerche » for a post-doctoral scholarship.

RIASSUNTO

Per poter discutere la validità del modello anarmonico di Einstein, come sviluppato da Henkel, è stato calcolato il rapporto tra le tensioni di vapore dei due isotopi del neon; nel calcolo si è seguito il metodo già adottato da Johns, usando però varie coppie di parametri ε e σ. I risultati sono confrontati con i recenti dati sperimentali di Roth e Bigeleisen. Si trova che la dipendenza dalla temperatura di $\ln p_m/p_M$ è poco sensibile alla scelta dei parametri molecolari ed è in buon accordo con i risultati sperimentali, ma meno di quanto dichiara il Johns. Si dimostra come la sola incertezza nella determinazione dei parametri molecolari non può render conto completamente del disaccordo tra dati teorici e sperimentali. Nello stesso schema si sono condotti i calcoli per l'argon. Si ottengono risultati numerici più bassi dei risultati sperimentali preliminari fornitici da Boato e Scoles. Si discute brevemente l'origine della differenza tra le tensioni di vapore misurate in isotopi separati e quelle dedotte mediante frazionamento in miscele isotopiche. Si suggerisce una possibile influenza della composizione isotopica della miscela solida sul fattore di frazionamento. Infine il $\ln p_m/p_M$ è calcolato per krypton e xenon mediante il teorema quantistico degli stati corrispondenti.

Millimetre Wave Spectrum of Deutero-Derivatives of Formic Acid (*).

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(ricevuto il 9 Luglio 1960)

Summary. — The rotational constant A of DCOOH, HCOOD and DCOOD have been determined by analysing the «a» type transitions in the range of frequencies from 120000 MHz to 160000 MHz. By assuming I_a and I_b as the equilibrium moments of inertia, the Kraitchman's method has been used to determine the co-ordinates of the two hydrogen atoms in the principal axis system of HCOOH. The centrifugal distortion constants of the three molecules are also given.

1. – Introduction.

Formic acid has been already studied by many investigators in microwave spectroscopy. Among them most recently Erlandsson [1] has been using millimetric techniques in order to study HCOOH and Lerner, Dailey and Friend [2] have been studying five isotopic species of the acid from 20000 to 50000 MHz. By using this range of frequencies it was impossible to determine the rotational constants A, and the molecular structure was obtained by use of the moments of inertia I_b and I_c: therefore the position of the hydrogen atom attached to the carbon could not be directly determined because of zero-point vibrations.

(*) This research was partly supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command.

(1) G. Erlandsson: *Journ. Chem. Phys.*, **28**, 71 (1958).

(2) G. Lerner, B. P. Dailey and J. P. Friend: *Journ. Chem. Phys.*, **26**, 680 (1957).

The present work is an attempt of getting a better molecular structure by studying the spectra of the deutero-derivatives of formic acid in the millimetre wave region; by examining millimetric transitions it is possible to determine the constant A for each molecular species. The lines used where « a » type transitions sufficiently sensitive to A; but a complete centrifugal distortion analysis has been necessary to obtain the constants A with a reasonably good accuracy.

2. – Spectrum.

The measurements have been done at room temperature with a video-type spectrometer using harmonic generator and millimetre wave detector of the type developed in the Laboratory of Duke University ([3]). In the harmonic generation a specially treated silicon crystal kindly supplied us by R.S. Ohl of Bell Telephone Laboratories, was used. The frequency measurements were made on the fundamental with a frequency standard monitored by Station WWV of the National Bureau of Standards.

DCOOD used was part of a sample kindly sent us by Dr. H. W. Morgan of Oak Ridge National Laboratory; a sample of DCOOH was prepared from DCOOD by equilibrating it with H_2O and HCOOD by equilibrating commercial formic acid with D_2O.

For the localization of the spectrum, the constants B and C found by Lerner, Dailey and Friend ([2]) were used together with an approximate value of A, calculated by using the planarity relation $I_a - I_b - I_c = 0$, valid for rigid rotors. The observed lines were all « a » type transitions; an attempt to find « b » type transitions was unsuccessful. In particular all the transitions $J = 6 \rightarrow 7$ with $\Delta K_1 = 0$ and $\Delta K_{-1} = 1$ for DCOOH and HCOOD and all the transitions $J = 5 \rightarrow 6$ of the same kind for DCOOD were measured. For these transitions a rather large contribution of centrifugal distortion effects is to be expected. On the other hand the constant A is very strongly dependent on a even small shift of frequency (0.2 MHz corresponds to an uncertainty of about 5 MHz in A); consequently a complete centrifugal distortion analysis was required in order that the uncertainty on A should be due only to the experimental error.

The maximum experimental error on the average value of several measurements of each frequency was 0.03 MHz on the fundamental and about 0.2 MHz on the transition frequencies.

([3]) W. C. King and W. Gordy: *Phys. Rev.*, **93**, 407 (1954).

3. – Centrifugal distortion analysis.

In order to determine the centrifugal distortion constants a convenient formula developed by JOHN G. BAKER [4] was used, based on the use of the reduced energy « w » and of the asymmetry parameter b. If we denote by W' the centrifugal distortion energy, the formula reads:

$$(1)\qquad W'/h = bR_{10}(J+1)J - \tfrac{5}{2}bwR_{10} - (D_J + \tfrac{1}{2}bR_{10})J^2(J+1)^2 - \\ - (D_{JK} - bR_{10})J(J+1)(w - b\,\mathrm{d}w/\mathrm{d}b) - 2\delta_J J(J+1)\,\mathrm{d}w/\mathrm{d}b + \\ + R_{10}w\,\mathrm{d}w/\mathrm{d}b + (R_6 - \tfrac{1}{4}bR_{10})T_{JK}\,,$$

with $R_{10} = (bD_K + 4R_5)/(1 - \frac{1}{2}b^2)$; all the constants are those defined by NIELSEN [5].

T_{JK} is important only in the following cases:

$$K = 0: \qquad T_{JK} = \tfrac{1}{16}b^2(J-3)(J-2)(J-1)J(J+1)(J+2)(J+3)(J+4)\,,$$

$$K = 1: \qquad T_{JK} = \frac{8c_1R_5}{b}\,(w - 1 - c_1b)\,,$$

$$K = 2: \qquad T_{JK} = \pm\,(J-1)\,J(J+1)(J+2) \qquad (E^{\pm})\,.$$

The order of magnitude of the constants was first evaluated by various approximate methods. Initial values for D_J, D_{JK} and δ_J were calculated by the sum rules [1] applied to the whole series of « a » type transitions.

The following two relations, valid for every molecule with a plane of symmetry, were also very helpful in the centrifugal analysis:

$$(2)\qquad P_5 = \tfrac{1}{2}(D_J + \delta_J - \tfrac{1}{2}D_{JK}) - (I_b^2/2I_a^2)(D_J + 6R_6) - (I_a^2/I_c^2)(D_J + D_{JK} + D_K) + \\ + (I_b^4/2I_a^2I_c^2)(D_J - 2\delta_J - 2R_6)\,,$$

$$(3)\qquad R_6 = \frac{[(I_c^2 - I_b^2)^2 - I_a^4]D_J - I_a^4(D_{JK} + D_K) + 2(I_c^4 - I_b^4)\delta_J}{2(I_c^4 + 6I_c^2I_b^2 + I_b^4)}\,.$$

A first approximate value for D_K was obtained by use of the relation: $D_J + D_{JK} + D_K \propto 1/I_a^4$ valid for a planar molecule, assuming the same proportionality constant as in HCOOH; in this case D_K was calculated by Erlandsson's data for D_J, D_{JK}, δ_J and R_6 using relation (3). R_5 and R_6 could be easily evaluated from (2) and (3). Final values for all the constants were obtained by an iteration method.

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(4) J. G. BAKER: *Bull. Am. Phys. Soc.*, **5**, 241 (1960); P. FAVERO, A. M. MIRRI e J. G. BAKER: *Nuovo Cimento*, **17**, 740 (1960).

(5) H. H. NIELSEN: *Rev. Mod. Phys.*, **23**, 90 (1951).

TABLE I. – *Frequency* (MHz).

Transition	Observed	Calculated	Centrifugal shift
		DCOOH	
$6_{06} \rightarrow 7_{07}$	150 345.06	150 345.07	— 8.76
$6_{15} \rightarrow 7_{16}$	160 392.07	160 392.12	—12.04
$6_{16} \rightarrow 7_{17}$	145 882.95	145 882.76	— 6.93
$6_{24} \rightarrow 7_{25}$	157 241.48	157 241.60	—10.10
$6_{25} \rightarrow 7_{26}$	153 521.68	153 521.47	— 7.82
$6_{33} \rightarrow 7_{34}$	154 757.18	154 757.27	— 6.09
$6_{34} \rightarrow 7_{35}$	154 571.17	154 570.95	— 5.87
$6_{42} \rightarrow 7_{43}$	154 431.13	154 431.16	— 2.26
$6_{43} \rightarrow 7_{44}$	154 427.84	154 427.91	— 2.40
$6_{5} \rightarrow 7_{5}$	154 332.94	154 333.11	+ 2.49
$6_{6} \rightarrow 7_{6}$	154 285.66	154 285.75	+ 8.29
$0_{00} \rightarrow 1_{01}$	22 011.3 (*a*)	22 011.47	— 0.03
$1_{10} \rightarrow 2_{11}$	46 122.6 (*a*)	46 122.95	— 0.25
$1_{11} \rightarrow 2_{12}$	41 923.0 (*a*)	41 922.71	— 0.30
$4_{14} \rightarrow 4_{13}$	20 989.6 (*a*)	20 989.82	— 0.75
		HCOOD	
$6_{06} \rightarrow 7_{07}$	149 753.66	149 753.58	—12.65
$6_{15} \rightarrow 7_{16}$	157 770.81	157 770.99	—16.26
$6_{16} \rightarrow 7_{17}$	145 312.14	145 312.30	—10.47
$6_{24} \rightarrow 7_{25}$	154 137.58	154 137.57	—12.64
$6_{25} \rightarrow 7_{26}$	151 778.74	151 778.60	—10.62
$6_{33} \rightarrow 7_{34}$	152 526.01	152 525.89	— 7.20
$6_{34} \rightarrow 7_{35}$	152 442.80	152 442.69	— 7.23
$6_{42} \rightarrow 7_{43}$	152 344.38	152 344.41	— 1.40
$6_{43} \rightarrow 7_{44}$	152 343.36	152 343.37	— 1.42
$6_{5} \rightarrow 7_{5}$	152 288.87	152 288.83	+ 5.82
$6_{6} \rightarrow 7_{6}$	152 264.95	152 265.14	+14.59
$0_{00} \rightarrow 1_{01}$	21 732.6 (*a*)	21 732.56	— 0.04
$1_{10} \rightarrow 2_{11}$	45 257.8 (*a*)	45 257.34	— 0.34
$1_{01} \rightarrow 2_{02}$	43 420.2 (*a*)	43 421.23	— 0.35
$1_{11} \rightarrow 2_{12}$	41 673.0 (*a*)	41 672.42	— 0.28
$5_{15} \rightarrow 5_{14}$	26 865.5 (*a*)	26 865.75	— 3.08

(*a*) Frequencies measured by LERNER, DAILEY and FRIEND.

TABLE I (*continued*).

Transition	Observed	Calculated	Centrifugal shift
	DCOOD		
$5_{05} \rightarrow 6_{06}$	124 685.34	124 685.16	— 5.73
$5_{14} \rightarrow 6_{15}$	133 623.69	133 623.70	— 8.67
$5_{15} \rightarrow 6_{66}$	120 407.60	120 407.47	— 4.36
$5_{23} \rightarrow 6_{24}$	130 411.85	130 411.68	— 7.11
$5_{24} \rightarrow 6_{25}$	127 342.80	127 342.91	— 5.40
$5_{32} \rightarrow 6_{33}$	128 336.34	128 336.39	— 4.28
$5_{33} \rightarrow 6_{34}$	128 203.04	128 203.19	— 4.13
$5_{41} \rightarrow 6_{42}$	128 077.64	128 077.74	— 1.57
$5_{42} \rightarrow 6_{43}$	128 075.90	128 075.82	— 1.43
$5_{5} \rightarrow 6_{5}$	127 996.80	127 996.65	— 1.90
$0_{00} \rightarrow 1_{01}$	21 293.9 (*a*)	21 293.93	— 0.02
$1_{10} \rightarrow 2_{11}$	44 813.4 (*a*)	44 813.22	— 0.33
$4_{14} \rightarrow 4_{13}$	22 237.5 (*a*)	22 237.63	— 1.99

(*a*) Frequencies measured by LERNER, DAILEY and FRIEND.

All the calculated transitions (Table I) are in very good agreement with the observed ones including those maesured by LERNER, DAILEY and FRIEND. For some of these, the $4_{14} \rightarrow 4_{13}$ of DCOOH and DCOOD and the $5_{15} \rightarrow 5_{14}$ of HCOOD the centrifugal distortion contribution is very considerable (of the

TABLE II. – *Rotational and centrifugal constants in* MHz.

	HCOOH (*a*)	DCOOH	HCOOD	DCOOD
A	77 520.07	57 710 ± 5	66 110 ± 5	50 815 ± 5
B	12 054.98	12 055.80	11 762.55	11 759.8
C	10 416.16	9 955.71	9 970.05	9 534.15
D_J	+0.009 28	+0.008 45	+0.010 42	+0.008 37
D_{JK}	—0.083 0	—0.036 6	—0.057 5	—0.030 9
D_K	+3.12	+1.21	+1.77	+0.75
δ_J	—0.002 02	—0.001 9	—0.002 1	—0.002 3
R_6	—0.000 23	—0.000 20	—0.000 18	—0.000 22
R_5	—	—0.004 0	+0.001 4	+0.006 5
R_{10}	—	—0.040	—0.023	+0.006

(*a*) ERLANDSSON [1].

order of 1 MHz), such effects being expecially due to δ_J and R_{10}; they have therefore been of help in the evaluation of R_{10}. In Table II are listed the rotational and centrifugal distortion constants of the various isotopic species of formic acid. The centrifugal distortion constants of each isotopic species are comparable in magnitude and of the same order as those determined by ERLANDSSON for HCOOH.

4. – Molecular structure.

The Kraitchman's method (6) of the isotopic substitution was used. By this method it should be possible to obtain exactly the co-ordinates of the substituted atom in the principal axis system of HCOOH if the equilibrium moments of inertia were known. Applying Kraitchman's method to the effective moments of inertia one obtains co-ordinates that are not the equilibrium ones but, as COSTAIN (7) shows, differ from them less than the r_0 defined by:

$$I_{\text{eff}} = \sum_i m_i r_{0i}^2 = \frac{1}{\left(1/\sum m_i r_i^2\right)_{\text{av}}},$$

where r_i are the instantaneous co-ordinate of the atom i and they are nearly independent on the isotopic species used. The combination HCOOH, HCOOD, DCOOH has been used for the localization of the two hydrogen atoms, taking I_a and I_c, among the three possible couples of moments of inertia, as those of equilibrium; in fact use of I_b gives an imaginary co-ordinate for the hydrogen atom attached to the carbon because it is very near to the b axis and the zero point vibrations make impossible the evaluation of the distance between the hydrogen and the b axis; using I_a and I_c this problem does not arise. For the localization of the C atom, I_b and I_c of HCOOH and $H^{13}COOH$ have been used as obtained by ERLANDSSON and DAILEY respectively; on the other hand, use of a particular couple of moments of inertia should be unimportant for the localization of this atom because it is very near to the center of gravity. For the calculation of the oxygen atoms co-ordinates have been used the equations for the center of gravity: $\sum m_i x_i = 0$, etc., and for the moments of inerta $\sum m_i x_i^2 = I_y$, etc. In Table III are listed the results obtained for the angles and bond lengths and the calculated rotational constants of DCOOD that were determined treating the molecule as a rigid rotor and subtracting from the calculated value of I_b the experimental inertial defect. Taking in

(6) J. KRAITCHMANN: *Am. Journ. Phys.*, **21**, 17 (1953).
(7) C. C. COSTAIN: *Journ. Chem. Phys.*, **29**, 864 (1958).

account the fact that DCOOD was not used for the determination of the molecular structure the agreement between the experimental and the calculated

TABLE III. – *Structural parameters.*

	This work	LERNER, DAILEY and FRIEND
C—H	$1.092A \pm 0.01A$	$1.085A \pm 0.025A$
C=O	$1.237A \pm 0.002A$	$1.245A \pm 0.002A$
C—O	$1.312A \pm 0.002A$	$1.312A \pm 0.002A$
O—H	$0.961A \pm 0.01A$	$0.95A \pm 0.020A$
< OCO	$124° 57' \pm 15'$	$124° 18' \pm 10'$
< HC=O	$120° 58' \pm 15'$	—
< COH	$170° 50' \pm 10'$	$107° 48' \pm 1°$

Rotational constants of DCOOD (MHz)

	Observed	Calculated
A	50 815 ± 5	50 840
B	11759.80	11758.44
C	9 534.15	9 534.07

rotational constants is satisfactorily good. The molecular structure obtained is in good agreement with the model D chosen by LERNER, DAILEY and FRIEND, except for the $C = 0$ bond length and the two HCO angles.

* * *

The author wishes to thank Dr. JOHN G. BAKER for many helpful suggestions during the analysis of experimental data and Dr. PAOLO G. FAVERO for his assistance in the early stage of the experimental work.

RIASSUNTO

Sono state determinate le costanti rotazionali A di DCOOH, HCOOD e DCOOD analizzando le transizioni di tipo « a » nell'intervallo di frequenze fra 120 000 MHz e 160 000 MHz. Assumendo I_a e I_b come momenti d'inerzia di equilibrio, è stato usato il metodo di Kraitchman per determinare le coordinate dei due atomi di idrogeno nel sistema degli assi principali di HCOOH. È stata anche fatta l'analisi degli effetti di distorsione centrifuga per le tre molecole.

Decay-Modes of the Schizon. An Isobaric Spin Analysis.

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(ricevuto il 26 Luglio 1960)

Summary. — An isobaric spin analysis of all possible decay-modes of the schizon into two and three strongly interacting particles has been made.

Recently it has been suggested (1) that the weak interactions are transmitted by an intermediate particle named the schizon having finite mass and spin one. The weak interaction Lagrangian, in this picture, consists of two terms in one of which the schizon exhibits an isospin triplet property and in the other a doublet property; total isobaric spin is conserved at each vortex. The predicted value for the mass of the schizon is uncertain. If the mass of the schizon is denoted by m_S, then the change in the Michel parameter ϱ for the μ-decay due to the transmitting field S is given by (1)

$$\varrho - 0.75 \simeq \frac{1}{3}\left(\frac{m_\mu}{m_S}\right)^2$$

The best experimental value for ϱ allowing for virtual radiative effects and inner bremsstrahlung, is 0.741 ± 0.027 (2). This gives a value for $m_S \cong$

(*) Also at Physics Department, University College, London.

(1) T. D. Lee and C. N. Yang: *Phys. Rev. Lett.*, **4**, 307 (1960); A. Salam and J. C. Ward: to be published in *Nuovo Cimento*; J. J. Sakurai: to be published in *Phys. Rev.*

(2) W. F. Dudziak, R. Sagane and J. Vedder: *Phys. Rev.*, **114**, 336 (1959).

$\simeq 1300\ m_e$; however, the upper limit is very uncertain owing to the quoted error in the value for ϱ.

It may be of interest, from the point of view of experimental detection of this particle, to examine all its possible modes of decay. In this note it is assumed that the schizon is sufficiently heavy to decay into a baryon pair plus a π- or a K-meson. An isobaric spin analysis of all possible modes of decay into two and three particles other than leptons has been made. The matrix elements for the different decay-modes of the S^+ and S^- are given in Table I. The case of the neutral S^0 has not been examined.

TABLE I-*a* (*). – S^+ *and* S^- *decays into three strongly interacting particles.*

$$S \to \pi + \pi + \pi$$

1. $\langle \pi^+ + \pi^+ + \pi^- | S^+ \rangle = \sqrt{\frac{3}{5}} (\pi)_1 = \langle \pi^- + \pi^- + \pi^+ | S^- \rangle$

2. $\langle \pi^0 + \pi^0 + \pi^+ | S^+ \rangle = \left(\frac{1}{\sqrt{15}} - \frac{1}{\sqrt{3}} \right) (\pi)_1 = \langle \pi^0 + \pi^0 + \pi^- | S^- \rangle$

$$S \to \pi + \pi + K$$

3. $\langle \pi^+ + \pi^- + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}} (\pi K)^1_{\frac{1}{2}} + \frac{1}{\sqrt{3}} (\pi K)^0_{\frac{1}{2}} = \langle \pi^- + \pi^+ + K^- | S^- \rangle$

4. $\langle \pi^+ + \pi^0 + K^0 | S^+ \rangle = \frac{1}{\sqrt{3}} (\pi K)^1_{\frac{1}{2}} = \langle \pi^- + \pi^0 + \overline{K^0} | S^- \rangle$

5. $\langle \pi^0 + \pi^0 + K^+ | S^+ \rangle = -\frac{1}{\sqrt{3}} (\pi K)^0_{\frac{1}{2}} = \langle \pi^0 + \pi^0 + K^- | S^- \rangle$

$$S \to K + K + \pi$$

6. $\langle K^+ + K^- + \pi^+ | S^+ \rangle = -\frac{1}{2} (K\pi)^1_1 + \frac{1}{\sqrt{2}} (K\pi)^0_1 = \langle K^0 + \overline{K^0} + \pi^- | S^- \rangle$

7. $\langle K^0 + \overline{K^0} + \pi^+ | S^+ \rangle = -\frac{1}{2} (K\pi)^1_1 - \frac{1}{\sqrt{2}} (K\pi)^0_1 = \langle K^+ + K^- + \pi^- | S^- \rangle$

8. $\langle K^+ + \overline{K^0} + \pi^0 | S^+ \rangle = \frac{1}{\sqrt{2}} (K\pi)^1_1 = \langle K^0 + K^- + \pi^0 | S^- \rangle$

(*) The notation $(\mathcal{N}\pi)^1_1$, *e.g.* denotes the amplitude when the nucleon and the antinucleon are in an isobaric spin one state and the final isobaric spin is one. The notation $(\Sigma\mathcal{N}\pi)^{\frac{3}{2}}_{\frac{1}{2}}$, *e.g.* denotes the amplitude when the sigma (antisigma) and the antinucleon (nucleon) are in an isobaric spin $\frac{3}{2}$ state and the final isobaric spin is $\frac{1}{2}$.

TABLE I-*a* (*continued*).

$$S \to K + K + K$$

9. $\langle K^+ + K^0 + \overline{K^0} | S^+ \rangle = \left(-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} \right) (K)_{\frac{1}{2}} = \langle K^- + \overline{K^0} + K^0 | S^- \rangle$

10. $\langle K^+ + K^- + K^- | S^+ \rangle = \left(-\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}} \right) (K)_{\frac{1}{2}} = \langle K^- + K^+ + K^- | S^- \rangle$

$$S \to \mathcal{N} + \mathcal{N} + \pi$$

11. $\langle P + \bar{P} + \pi^+ | S^+ \rangle = -\frac{1}{2} (\mathcal{N}\pi)_1^1 + \frac{1}{\sqrt{2}} (\mathcal{N}\pi)_1^0 = \langle N + \bar{N} + \pi^- | S^- \rangle$

12. $\langle N + \bar{N} + \pi^+ | S^+ \rangle = -\frac{1}{2} (\mathcal{N}\pi)_1^1 - \frac{1}{\sqrt{2}} (\mathcal{N}\pi)_1^0 = \langle P + \bar{P} + \pi^- | S^- \rangle$

13. $\langle P + \bar{N} + \pi^0 | S^+ \rangle = \frac{1}{\sqrt{2}} (\mathcal{N}\pi)_1^1 = \langle N + \bar{P} + \pi^0 | S^- \rangle$

$$S \to \mathcal{N} + \mathcal{N} + K$$

14. $\langle P + \bar{P} + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}} (\mathcal{N}K)_{\frac{1}{2}}^{\frac{1}{2}} + \frac{1}{\sqrt{2}} (\mathcal{N}K)_{\frac{1}{2}}^0 = \langle \bar{P} + P + K^- | S^- \rangle$

15. $\langle N + \bar{N} + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}} (\mathcal{N}K)_{\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{\sqrt{2}} (\mathcal{N}K)_{\frac{1}{2}}^0 = \langle \bar{N} + N + K^- | S^- \rangle$

16. $\langle P + \bar{N} + K^0 | S^+ \rangle = \sqrt{\frac{2}{3}} (\mathcal{N}K)_{\frac{1}{2}}^{\frac{1}{2}} = \langle \bar{P} + N + \overline{K^0} | S^- \rangle$

$$S \to \Lambda + \mathcal{N} + \pi$$

17. $\langle \overline{\Lambda^0} + P + \pi^0 | S^+ \rangle = -\frac{1}{\sqrt{3}} (\Lambda\mathcal{N}\pi)_{\frac{1}{2}} = \langle \Lambda^0 + \bar{P} + \pi^0 | S^- \rangle$

18. $\langle \overline{\Lambda^0} + N + \pi^+ | S^+ \rangle = -\sqrt{\frac{2}{3}} (\Lambda\mathcal{N}\pi)_{\frac{1}{2}} = \langle \Lambda^0 + \bar{N} + \pi^- | S^- \rangle$

$$S \to \Lambda + \mathcal{N} + K$$

19. $\langle \Lambda^0 + \bar{N} + K^+ | S^+ \rangle = (\Lambda\mathcal{N}K)_1 = \langle \Lambda^0 + \bar{P} + K^0 | S^- \rangle$

20. $\langle \Lambda^0 + P + K^0 | S^- \rangle = (\Lambda\mathcal{N}K)_1 = \langle \overline{\Lambda^0} + N + K^- | S^- \rangle$

$$S \to \Lambda + \Lambda + \pi$$

21. $\langle \Lambda^0 + \overline{\Lambda^0} + \pi^+ | S^+ \rangle = (\Lambda\Lambda\pi)_1 = \langle \Lambda^0 + \overline{\Lambda^0} + \pi^- | S^- \rangle$

$$S \to \Lambda + \Lambda + K$$

22. $\langle \Lambda^0 + \overline{\Lambda^0} + K^+ | S^+ \rangle = (\Lambda\Lambda K)_{\frac{1}{2}} = \langle \overline{\Lambda^0} + \Lambda^0 + K^- | S^- \rangle$

TABLE I-*a* (*continued*).

$$S \to \Sigma + \mathcal{N} + \pi$$

23. $\langle \overline{\Sigma^+} + P + \pi^+ | S^+ \rangle = -\frac{1}{\sqrt{18}} (\Sigma\mathcal{N}\pi)^{\frac{3}{2}}_{\frac{3}{2}} + \frac{2}{3} (\Sigma\mathcal{N}\pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Sigma^+ + \overline{P} + \pi^- | S^- \rangle$

24. $\langle \overline{\Sigma^0} + P + \pi^0 | S^+ \rangle = -\frac{\sqrt{2}}{3} (\Sigma\mathcal{N}\pi)^{\frac{3}{2}}_{\frac{1}{2}} - \frac{1}{3} (\Sigma\mathcal{N}\pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Sigma^0 + \overline{P} + \pi^0 | S^- \rangle$

25. $\langle \overline{\Sigma^0} + N + \pi^+ | S^+ \rangle = \frac{1}{3} (\Sigma\mathcal{N}\pi)^{\frac{3}{2}}_{\frac{1}{2}} - \frac{\sqrt{2}}{3} (\Sigma\mathcal{N}\pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Sigma^0 + \overline{N} + \pi^- | S^- \rangle$

26. $\langle \overline{\Sigma^-} + P + \pi^- | S^+ \rangle = \frac{1}{\sqrt{2}} (\Sigma\mathcal{N}\pi)^{\frac{3}{2}}_{\frac{3}{2}} = \langle \Sigma^- + \overline{P} + \pi^+ | S^- \rangle$

27. $\langle \overline{\Sigma^-} + N + \pi^0 | S^- \rangle = -\frac{1}{3} (\Sigma\mathcal{N}\pi)^{\frac{3}{2}}_{\frac{1}{2}} + \frac{\sqrt{2}}{3} (\Sigma\mathcal{N}\pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Sigma^- + \overline{N} + \pi^0 | S^- \rangle$

$$S \to \Sigma + \mathcal{N} + K$$

28. $\langle \Sigma^0 + \overline{N} + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}} (\Sigma\mathcal{N}K)^{\frac{3}{2}}_{1} - \frac{1}{\sqrt{3}} (\Sigma\mathcal{N}K)^{\frac{1}{2}}_{1} = \langle \Sigma^0 + \overline{P} + K^0 | S^- \rangle$

29. $\langle \overline{\Sigma^0} + P + \overline{K^0} | S^+ \rangle = -\frac{1}{\sqrt{6}} (\Sigma\mathcal{N}K)^{\frac{3}{2}}_{1} - \frac{1}{\sqrt{3}} (\Sigma\mathcal{N}K)^{\frac{1}{2}}_{1} = \langle \overline{\Sigma^0} + N + K^- | S^- \rangle$

30. $\langle \Sigma^+ + \overline{P} + K^- | S^+ \rangle = -\frac{1}{2\sqrt{3}} (\Sigma\mathcal{N}K)^{\frac{3}{2}}_{1} + \sqrt{\frac{2}{3}} (\Sigma\mathcal{N}K)^{\frac{1}{2}}_{1} = \langle \Sigma^- + \overline{N} + K^0 | S^- \rangle$

31. $\langle \overline{\Sigma^-} + N + \overline{K^0} | S^+ \rangle = -\frac{1}{2\sqrt{3}} (\Sigma\mathcal{N}K)^{\frac{3}{2}}_{1} + \sqrt{\frac{2}{3}} (\Sigma\mathcal{N}K)^{\frac{1}{2}}_{1} = \langle \overline{\Sigma^+} + P + K^- | S^- \rangle$

32. $\langle \Sigma^+ + \overline{N} + K^0 | S^+ \rangle = \frac{\sqrt{3}}{2} (\Sigma\mathcal{N}K)^{\frac{3}{2}}_{1} = \langle \Sigma^- + \overline{P} + K^+ | S^- \rangle$

33. $\langle \overline{\Sigma^-} + P + K^- | S^+ \rangle = \frac{\sqrt{3}}{2} (\Sigma\mathcal{N}K)^{\frac{3}{2}}_{1} = \langle \overline{\Sigma^+} + N + \overline{K^0} | S^- \rangle$

$$S \to \Sigma + \Lambda + \pi$$

34. $\langle \Sigma^0 + \Lambda^0 + \pi^+ | S^+ \rangle = -\frac{1}{\sqrt{2}} (\Sigma\Lambda\pi)_1 = \langle \overline{\Sigma^0} + \Lambda^0 + \pi^- | S^- \rangle$

35. $\langle \Sigma^0 + \overline{\Lambda^0} + \pi^+ | S^+ \rangle = -\frac{1}{\sqrt{2}} (\Sigma\Lambda\pi)_1 = \langle \Sigma^0 + \overline{\Lambda^0} + \pi^- | S^- \rangle$

36. $\langle \Sigma^+ + \overline{\Lambda^0} + \pi^0 | S^+ \rangle = \frac{1}{\sqrt{2}} (\Sigma\Lambda\pi)_1 = \langle \Sigma^- + \overline{\Lambda^0} + \pi^0 | S^- \rangle$

37. $\langle \overline{\Sigma^-} + \Lambda^0 + \pi^0 | S^+ \rangle = \frac{1}{\sqrt{2}} (\Sigma\Lambda\pi)_1 = \langle \overline{\Sigma^+} + \Lambda^0 + \pi^0 | S^- \rangle$

$$S \to \Sigma + \Lambda + K$$

38. $\langle \Sigma^+ + \overline{\Lambda^0} + \overline{K^0} | S^+ \rangle = \sqrt{\frac{2}{3}} (\Sigma\Lambda K)_{\frac{1}{2}} = \langle \overline{\Sigma^+} + \Lambda^0 + \overline{K^0} | S^- \rangle$

TABLE I-*a* (*continued*).

39. $\langle \Sigma^0+\overline{\Lambda^0}+K^+|S^+\rangle = -\frac{1}{\sqrt{3}}(\Sigma\Lambda K)_{\frac{1}{2}} = \langle \overline{\Sigma^0}+\Lambda^0+K^-|S^-\rangle$

40. $\langle \overline{\Sigma^0}+\Lambda^0+K^+|S^+\rangle = -\frac{1}{\sqrt{3}}(\Sigma\Lambda K)_{\frac{1}{2}} = \langle \Sigma^0+\overline{\Lambda^0}+K^-|S^-\rangle$

41. $\langle \overline{\Sigma^-}+\Lambda^0+K^0|S^+\rangle = \sqrt{\frac{2}{3}}(\Sigma\Lambda K)_{\frac{1}{2}} = \langle \Sigma^-+\overline{\Lambda^0}+\overline{K^0}|S^-\rangle$

$S \to \Sigma+\Sigma+\pi$

42. $\langle \Sigma^+ + \overline{\Sigma^0}+\pi^0|S^+\rangle = -\sqrt{\frac{3}{20}}(\Sigma\pi)^2_1 + \frac{1}{2}(\Sigma\pi)^1_1 = \langle \Sigma^- + \overline{\Sigma^0}+\pi^0|S^-\rangle$

43. $\langle \Sigma^0+\overline{\Sigma^0}+\pi^+|S^+\rangle = \frac{1}{\sqrt{15}}(\Sigma\pi)^2_1 - \frac{1}{\sqrt{3}}(\Sigma\pi)^0_1 = \langle \Sigma^0+\overline{\Sigma^0}+\pi^-|S^-\rangle$

44. $\langle \Sigma^0+\overline{\Sigma^-}+\pi^0|S^+\rangle = -\sqrt{\frac{3}{20}}(\Sigma\pi)^2_1 - \frac{1}{2}(\Sigma\pi)^1_1 = \langle \Sigma^0+\overline{\Sigma^+}+\pi^0|S^-\rangle$

45. $\langle \Sigma^-+\overline{\Sigma^-}+\pi^+|S^+\rangle = \frac{1}{\sqrt{60}}(\Sigma\pi)^2_1 - \frac{1}{2}(\Sigma\pi)^1_1 + \frac{1}{\sqrt{3}}(\Sigma\pi)^0_1 = \langle \Sigma^+ + \overline{\Sigma^+}+\pi^-|S^-\rangle$

46. $\langle \Sigma^+ + \overline{\Sigma^+}+\pi^+|S^+\rangle = \frac{1}{\sqrt{60}}(\Sigma\pi)^2_1 - \frac{1}{2}(\Sigma\pi)^1_1 + \frac{1}{\sqrt{3}}(\Sigma\pi)^0_1 = \langle \Sigma^- + \overline{\Sigma^+}+\pi^-|S^-\rangle$

47. $\langle \Sigma^+ + \overline{\Sigma^-}+\pi^-|S^+\rangle = \sqrt{\frac{3}{5}}(\Sigma\pi)^2_1 = \langle \Sigma^- + \overline{\Sigma^+}+\pi^+|S^-\rangle$

$S \to \Sigma+\Sigma+K$

48. $\langle \Sigma^+ + \overline{\Sigma^0}+K^0|S^+\rangle = \frac{1}{\sqrt{3}}(\Sigma K)^1_{\frac{1}{2}} = \langle \overline{\Sigma^+}+\Sigma^0+\overline{K^0}|S^-\rangle$

49. $\langle \Sigma^0+\overline{\Sigma^0}+K^+|S^+\rangle = -\frac{1}{\sqrt{3}}(\Sigma K)^0_{\frac{1}{2}} = \langle \overline{\Sigma^0}+\Sigma^0+K^-|S^-\rangle$

50. $\langle \Sigma^0+\overline{\Sigma^-}+K^0|S^+\rangle = -\frac{1}{\sqrt{3}}(\Sigma K)^1_{\frac{1}{2}} = \langle \overline{\Sigma^0}+\Sigma^-+\overline{K^0}|S^-\rangle$

51. $\langle \Sigma^-+\overline{\Sigma^-}+K^+|S^+\rangle = \frac{1}{\sqrt{6}}(\Sigma K)^1_{\frac{1}{2}} + \frac{1}{\sqrt{3}}(\Sigma K)^0_{\frac{1}{2}} = \langle \overline{\Sigma^-}+\Sigma^-+K^-|S^-\rangle$

52. $\langle \Sigma^+ + \overline{\Sigma^+}+K^+|S^+\rangle = -\frac{1}{\sqrt{6}}(\Sigma K)^1_{\frac{1}{2}} + \frac{1}{\sqrt{3}}(\Sigma K)^0_{\frac{1}{2}} = \langle \overline{\Sigma^+}+\Sigma^+ + K^-|S^-\rangle$

$S \to \Xi+\Lambda+\pi$

53. $\langle \overline{\Xi^-}+\Lambda^0+\pi^0|S^+\rangle = -\frac{1}{\sqrt{3}}(\Xi\Lambda\pi)_{\frac{1}{2}} = \langle \Xi^-+\overline{\Lambda^0}+\pi^0|S^-\rangle$

54. $\langle \overline{\Xi^0}+\Lambda^0+\pi^+|S^+\rangle = \sqrt{\frac{2}{3}}(\Xi\Lambda\pi)_{\frac{1}{2}} = \langle \Xi^0+\overline{\Lambda^0}+\pi^-|S^-\rangle$

TABLE I-*a* (*continued*).

$$S \to \Xi + \mathcal{N} + K$$

55. $\langle \overline{\Xi^-} + N + \overline{K^0} | S^+ \rangle = -\frac{1}{\sqrt{6}} (\Xi \mathcal{N} K)^1_{\frac{1}{2}} + \frac{1}{\sqrt{2}} (\Xi \mathcal{N} K)^0_{\frac{1}{2}} = \langle \Xi^- + \overline{N} + K^0 | S^- \rangle$

56. $\langle \overline{\Xi^-} + P + K^- | S^+ \rangle = \sqrt{\frac{2}{3}} (\Xi \mathcal{N} K)^1_{\frac{1}{2}} = \langle \Xi^- + \overline{P} + K^+ | S^- \rangle$

57. $\langle \overline{\Xi^0} + P + \overline{K^0} | S^+ \rangle = -\frac{1}{\sqrt{6}} (\Xi \mathcal{N} K)^1_{\frac{1}{2}} - \frac{1}{\sqrt{2}} (\Xi \mathcal{N} K)^0_{\frac{1}{2}} = \langle \Xi^0 + \overline{P} + K^0 | S^- \rangle$

$$S \to \Xi + \Lambda + K$$

58. $\langle \overline{\Xi^-} + \Lambda^0 + \overline{K^0} | S^+ \rangle = (\Xi \Lambda K)_1 = \langle \overline{\Xi^0} + \Lambda^0 + K^- | S^- \rangle$

59. $\langle \Xi^0 + \overline{\Lambda^0} + K^+ | S^+ \rangle = (\Xi \Lambda K)_1 = \langle \Xi^- + \overline{\Lambda^0} + K^0 | S^- \rangle$

$$S \to \Xi + \Sigma + \pi$$

60. $\langle \overline{\Xi^-} + \Sigma^- + \pi^+ | S^+ \rangle = \frac{1}{\sqrt{18}} (\Xi \Sigma \pi)^{\frac{3}{2}}_{\frac{1}{2}} + \frac{2}{3} (\Xi \Sigma \pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Xi^- + \overline{\Sigma^-} + \pi^- | S^- \rangle$

61. $\langle \overline{\Xi^-} + \Sigma^0 + \pi^0 | S^+ \rangle = -\frac{\sqrt{2}}{3} (\Xi \Sigma \pi)^{\frac{3}{2}}_{\frac{1}{2}} - \frac{1}{3} (\Xi \Sigma \pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Xi^- + \overline{\Sigma^0} + \pi^0 | S^- \rangle$

62. $\langle \overline{\Xi^-} + \Sigma^+ + \pi^- | S^+ \rangle = \frac{1}{\sqrt{2}} (\Xi \Sigma \pi)^{\frac{3}{2}}_{\frac{1}{2}} = \langle \Xi^- + \overline{\Sigma^+} + \pi^+ | S^- \rangle$

63. $\langle \overline{\Xi^0} + \Sigma^+ + \pi^0 | S^+ \rangle = -\frac{1}{3} (\Xi \Sigma \pi)^{\frac{3}{2}}_{\frac{1}{2}} + \frac{\sqrt{2}}{3} (\Xi \Sigma \pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Xi^0 + \overline{\Sigma^+} + \pi^0 | S^- \rangle$

64. $\langle \overline{\Xi^0} + \Sigma^0 + \pi^+ | S^+ \rangle = \frac{1}{3} (\Xi \Sigma \pi)^{\frac{3}{2}}_{\frac{1}{2}} - \frac{\sqrt{2}}{3} (\Xi \Sigma \pi)^{\frac{1}{2}}_{\frac{1}{2}} = \langle \Xi^0 + \overline{\Sigma^0} + \pi^- | S^- \rangle$

$$S \to \Xi + \Sigma + K$$

65. $\langle \overline{\Xi^-} + \Sigma^0 + \overline{K^0} | S^+ \rangle = -\frac{1}{\sqrt{6}} (\Xi \Sigma K)^{\frac{3}{2}}_{1} - \frac{1}{\sqrt{3}} (\Xi \Sigma K)^{\frac{1}{2}}_{1} = \langle \overline{\Xi^0} + \Sigma^0 + K^- | S^- \rangle$

66. $\langle \overline{\Xi^0} + \Sigma^+ + \overline{K^0} | S^+ \rangle = -\frac{1}{2\sqrt{3}} (\Xi \Sigma K)^{\frac{3}{2}}_{1} + \sqrt{\frac{2}{3}} (\Xi \Sigma K)^{\frac{1}{2}}_{1} = \langle \overline{\Xi^-} + \Sigma^- + K^- | S^- \rangle$

67. $\langle \Xi^0 + \overline{\Sigma^0} + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}} (\Xi \Sigma K)^{\frac{3}{2}}_{1} - \frac{1}{\sqrt{3}} (\Xi \Sigma K)^{\frac{1}{2}}_{1} = \langle \Xi^- + \overline{\Sigma^0} + K^0 | S^- \rangle$

68. $\langle \overline{\Xi^-} + \Sigma^+ + K^- | S^+ \rangle = \frac{\sqrt{3}}{2} (\Xi \Sigma K)^{\frac{3}{2}}_{1} = \langle \overline{\Xi^0} + \Sigma^- + \overline{K^0} | S^- \rangle$

69. $\langle \Xi^0 + \overline{\Sigma^-} + K^0 | S^+ \rangle = \frac{\sqrt{3}}{2} (\Xi \Sigma K)^{\frac{3}{2}}_{1} = \langle \Xi^- + \overline{\Sigma^+} + K^+ | S^- \rangle$

$$S \to \Xi + \Xi + \pi$$

70. $\langle \Xi^0 + \overline{\Xi^0} + \pi^+ | S^+ \rangle = -\frac{1}{2} (\Xi \pi)^1_1 + \frac{1}{\sqrt{2}} (\Xi \pi)^0_1 = \langle \Xi^- + \overline{\Xi^-} + \pi^- | S^- \rangle$

TABLE I-*a* (*continued*).

71. $\langle \Xi^- + \overline{\Xi^-} + \pi^+ | S^+ \rangle = -\frac{1}{2}(\Xi\pi)_1^1 - \frac{1}{\sqrt{2}}(\Xi\pi)_1^0 = \langle \Xi^0 + \overline{\Xi^0} + \pi^- | S^- \rangle$

72. $\langle \Xi^0 + \overline{\Xi^-} + \pi^0 | S^+ \rangle = \frac{1}{\sqrt{2}}(\Xi\pi)_1^1 = \langle \Xi^- + \overline{\Xi^0} + \pi^0 | S^- \rangle$

$$S \to \Xi + \Xi + K$$

73. $\langle \Xi^0 + \overline{\Xi^0} + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}}(\Xi K)_{\frac{1}{2}}^1 + \frac{1}{\sqrt{2}}(\Xi K)_{\frac{1}{2}}^0 = \langle \overline{\Xi^0} + \Xi^0 + K^- | S^- \rangle$

74. $\langle \Xi^- + \overline{\Xi^-} + K^+ | S^+ \rangle = -\frac{1}{\sqrt{6}}(\Xi K)_{\frac{1}{2}}^1 - \frac{1}{\sqrt{2}}(\Xi K)_{\frac{1}{2}}^0 = \langle \overline{\Xi^-} + \Xi^- + K^- | S^- \rangle$

TABLE I-*b*. – S^+ *and* S^- *decays into two strongly interacting particles.*

75. $\langle \pi^+ + \pi^0 | S^+ \rangle = \frac{1}{\sqrt{2}}(\pi)_1 = \langle \pi^- + \pi^0 | S^- \rangle$

76. $\langle K^+ + \pi^0 | S^+ \rangle = -\frac{1}{\sqrt{3}}(K\pi)_{\frac{1}{2}} = \langle K^- + \pi^0 | S^- \rangle$

77. $\langle K^0 + \pi^+ | S^+ \rangle = \frac{1}{\sqrt{3}}(K\pi)_{\frac{1}{2}} = \langle \overline{K^0} + \pi^- | S^- \rangle$

78. $\langle K^+ + \overline{K^0} | S^+ \rangle = (K)_1 = \langle K^0 + K^- | S^- \rangle$

79. $\langle P + \bar{N} | S^+ \rangle = (\mathcal{N})_1 = \langle N + \bar{P} | S^- \rangle$

80. $\langle \overline{\Lambda^0} + P | S^+ \rangle = (\Lambda\mathcal{N})_{\frac{1}{2}} = \langle \Lambda^0 + \bar{P} | S^- \rangle$

81. $\langle \overline{\Sigma^-} + N | S^+ \rangle = \sqrt{\frac{2}{3}}(\Sigma\mathcal{N})_{\frac{1}{2}} = \langle \Sigma^- + \bar{N} | S^- \rangle$

82. $\langle \overline{\Sigma^0} + P | S^+ \rangle = -\frac{1}{\sqrt{3}}(\Sigma\mathcal{N})_{\frac{1}{2}} = \langle \Sigma^0 + \bar{P} | S^- \rangle$

83. $\langle \overline{\Sigma^-} + \Lambda^0 | S^+ \rangle = (\Sigma\Lambda)_1 = \langle \overline{\Sigma^+} + \Lambda^0 | S^- \rangle$

84. $\langle \Sigma^+ + \overline{\Lambda^0} | S^+ \rangle = (\Sigma\Lambda)_1 = \langle \Sigma^- + \overline{\Lambda^0} | S^- \rangle$

85. $\langle \Sigma^0 + \overline{\Sigma^-} | S^+ \rangle = -\frac{1}{\sqrt{2}}(\Sigma)_1 = \langle \Sigma^0 + \overline{\Sigma^+} | S^- \rangle$

86. $\langle \Sigma^+ + \overline{\Sigma^0} | S^+ \rangle = \frac{1}{\sqrt{2}}(\Sigma)_1 = \langle \Sigma^- + \overline{\Sigma^0} | S^- \rangle$

87. $\langle \overline{\Xi^-} + \Lambda^0 | S^+ \rangle = (\Xi\Lambda)_{\frac{1}{2}} = \langle \Xi^- + \overline{\Lambda^0} | S^- \rangle$

88. $\langle \overline{\Xi^0} + \Sigma^+ | S^+ \rangle = \sqrt{\frac{2}{3}}(\Xi\Sigma)_{\frac{1}{2}} = \langle \Xi^0 + \overline{\Sigma^+} | S^- \rangle$

89. $\langle \overline{\Xi^-} + \Sigma^0 | S^+ \rangle = -\frac{1}{\sqrt{3}}(\Xi\Sigma)_{\frac{1}{2}} = \langle \Xi^- + \overline{\Sigma^0} | S^- \rangle$

90. $\langle \Xi^0 + \overline{\Xi^-} | S^+ \rangle = (\Xi)_1 = \langle \Xi^- + \overline{\Xi^0} | S^- \rangle$

Due to more available phase space, the two-body decays will be more favoured—especially decays into leptons $e+\nu$ and $\mu+\nu$. The decay-rates of these latter reactions have been given by LEE and YANG [1].

* * *

I am greatly indebted to Professor ABDUS SALAM for valuable advice. Thanks are due to Mr. A. GHANI for working with me on this problem.

RIASSUNTO (*)

Si è effettuata un'analisi dello spin isobarico di tutti i possibili modi di decadimento dello schizone in due o tre particelle fortemente interagenti.

(*) *Traduzione a cura della Redazione.*

On the Total Specific Ionization.

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(ricevuto il 29 Luglio 1960)

Summary. — A method is given for calculating the absolute value of primary, secondary and total specific ionization generated by a ionizing particle traversing a given medium. Density effect is taken into account. Numerical results are given for H and He and compared with the existing experimental data. It is shown that primary ionization presents higher relativistic increase (lower density effect), than the total one where the diluition of secondary ionization is felt. Further primary ionization is independent of maximum transferable energy in close collisions which on the contrary influences sensibly ionization of higher generations. Simplified formulae are given where the influence of the physical characteristic of the medium on the phenomenon appears evident and which are apt to be compared with the experimental results.

1. – Introduction.

In recent times, renewed attention has been devoted [1] to the relativistic increase of specific ionization generated by energetic charged particles traversing matter, in view of the possibility of using this effect as a means of distinguishing the masses of particles having velocities very near that one of light.

These experiments have shown that the theory of such an effect is far from satisfactory. This fact is not surprising considering that what one usually does, is to compare effects due to the number of ionization events, with the

[1] R. G. Kepler, C. A. Andlau, W. B. Fretter and L. F. Hansen: *Nuovo Cimento*, **7**, 71 (1958); A. Rousset, A. Lagarrigue, P. Musset, P. Rançon and X. Sauteron: *Nuovo Cimento*, **14**, 365 (1959).

formula giving the energy loss suffered by the ionizing particles. In order to do this, one introduces in the theoretical expression of the total energy loss three parameters (mean ionization potential, maximum transferable energy and the mean energy spent in order to produce a pair of ions) more or less arbitrarily in order to fit the theoretical curves with the experimental data. But, there is also disagreements between the various experimental results found by the different authors and this fact should be interpreted as due to a rather pronunced dependence of the physical results on the particular experimental situation. A theory with empirical fitting parameters is certainly not the most apt to allow an insight into such a dependence.

We have thus attempted to develop a previous work [2] on the theoretical calculation of the absolute value of the primary specific ionization, taking into account the exact dispersive and absorptive properties of the medium and avoiding then the introduction of arbitrary parameters.

Considering that often what one measures is the effect not of primary but of total specific ionization, in Section **2** we give a general method for calculating such a quantity. In order to apply our theory to the H and Hs atoms, in Section **3** we have reported the expressions of the dielectric constante for these two media.

In Section **4** the formulae for the primary, secondary and tertiary specific ionization deduced in Section **2** has been applied to H and He, and the numerical theoretical results have been discussed together with the experimental data.

The expressions of our final results are rather complicated, because, due to the complicated expression of the dielectric constant, the integrals cannot be performed analitically.

Therefore, in Section **5** we deduced a simplified formula which, differing by less than 2% from the exact one, shows in a simple form the dependence of the theoretical results on the physical parameters and should be easily adapted to the different experimental situations. Another approximated method for evaluating the total specific ionization is also reported in this section. Then a final discussion of our results has been made in Section **6**.

2. – Specific total ionization.

Be $\partial N_1(E)/\partial z$ the mean specific primary ionization generated by a particle of energy E in a medium of atomic number Z.

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[2] P. Budini and L. Taffara: *Nuovo Cimento*, **4**, 23 (1956). In the following this work will be indicated with (I).

Let us write $\partial N_1(E)/\partial z$ in the form:

$$\frac{\partial N_1(E)}{\partial z} = \sum_{i=1}^{Z} \int_0^\infty \frac{\partial n_{1i}(\varepsilon, E)}{\partial z} \, \mathrm{d}\varepsilon \,, \tag{1}$$

where $\sum_{n=1}^{Z} n_{1i}(\varepsilon, E)$ represents the energy-spectrum of the electrons emitted from the ionized atoms. Moreover, be $N_T^e(\varepsilon)$ the total number of ions generated by an electron of energy ε during its way through the medium where it stops; the mean specific ionization generated by a particle of energy E is then given by

$$\frac{\partial N_T(E)}{\partial z} = \sum_{i=1}^{Z} \int_0^\infty \left(1 + N_T^e(\varepsilon)\right) \frac{\partial n_{1i}(\varepsilon, E)}{\partial z} \, \mathrm{d}\varepsilon \,. \tag{2}$$

If the primary particle is an electron, (2) represents an integro-differential equation of the unknown function $N_T^e(E)$.

Now, $N_T^e(E)$ can be written in the form

$$N_T^e(E) = \sum_{n=1}^{n_{\max}} N_n^e(E) \,, \tag{3}$$

where $N_n(E)$ is the number of ions of n-th generation created by an electron of energy E which stops in the medium, and $n_{\max}$ is the generation which produces electrons with energy certainly less than the minimum ionization potential. Substituting (3) in (2) one obtains

$$\frac{\partial N_T(E)}{\partial z} = \frac{\partial N_n(E)}{\partial z} + \sum_{n=1}^{n_{\max}} \frac{\partial N_{n+1}(E)}{\partial z} \,, \tag{4}$$

where

$$\frac{\partial N_{n+1}(E)}{\partial z} = \sum_{i=1}^{Z} \int_0^\infty N_n^e(\varepsilon) \frac{\partial n_{1i}(\varepsilon, E)}{\partial z} \, \mathrm{d}\varepsilon \,, \tag{5}$$

obviously represents the mean specific number of ions of $(n+1)$-th generation produced by the primary particle.

Owing to the physical meaning of $N_T^e(E)$, one can write:

$$N_T^e(E) = \int_0^R \frac{\partial N_T^e(E)}{\partial z} \, \mathrm{d}z = -\int_{E_{\min}}^{E} \frac{\partial N_T^e(E)}{\partial z} \cdot \left(\frac{\partial E}{\partial z}\right)^{-1} \mathrm{d}E \,, \tag{6}$$

where R and $-\partial E/\partial z$ are the range and the total specific energy loss of the electron of energy E, and $I_{\min}$ is the lowest ionization potential of the medium.

Similarly

$$N_n^e(E) = -\sum_{i=1}^{Z} \int_{nI_i}^{E} \frac{\partial N_{ni}^e(E)}{\partial z} \cdot \left(\frac{\partial E}{\partial z}\right)^{-1} \mathrm{d}E \,, \tag{7}$$

where

$$\frac{\partial N_{ni}^e(E)}{\partial z} = \int_0^{\infty} N_T^e(\varepsilon) \, \frac{\partial n_{Ii}^e(\varepsilon, E)}{\partial z} \, \mathrm{d}\varepsilon \qquad i = 1, 2, \ldots . \tag{8}$$

The condition

$$N_n^e(E) \equiv 0 \qquad \text{for} \qquad nI_{\min} \geqslant E$$

clearly defines the

$$n_{\max} = \frac{E}{I_{\min}} \,.$$

The total specific ionization can be calculated from (2) by means of (4) and (5) with $N_n^e(E)$ given by (7).

Alternatively introducing

$$w(E) = -\left(\frac{\partial E}{\partial z}\right) \cdot \left(\frac{\partial N_T^e(E)}{\partial z}\right)^{-1} = \text{mean energy necessary for producing an ion-pair} \,, \tag{9}$$

(7) becomes

$$N_T^e(E) = \int_{I_{\min}}^{E} \frac{\mathrm{d}E}{w(E)} \,, \tag{10}$$

which, inserted in (2), allows to calculate the total specific ionization if $w(E)$ is known.

3. – Dielectric constant and photoelectric cross-sections for H and He.

For the theoretical calculation of $\partial N_T(E)/\partial z$ for H and He we will assume, according to (I), the following expression for the dielectric constant of the medium

$$\varepsilon(x) = 1 + C \, \frac{n}{I^2} \left[\sum \frac{f_i}{x_i^2 - x^2 + i g_i x} + \int_1^{\infty} \frac{f(x') \, \mathrm{d}x'}{x'^2 - x^2 + i g(x) x} \right] , \tag{11}$$

where: f_i = oscillator-strength for the discrete electronic transitions,

$f_0 = \int_1^\infty f(x)\,\mathrm{d}x$ = oscillator-strength for the electronic transitions in the continuous spectrum,

x_i = frequencies of the absorption line measured in ω_0 (ionization frequency) units,

g_i = width of the absorption line in ω_0 units,

$$g(x) = \frac{2I^2x^2}{3mc^3\hbar^2}, \qquad x = \frac{\omega}{\omega_0},$$

and

$$C = \frac{4\pi e^2\hbar^2}{m}, \tag{12}$$

with n = number of electron for cm^3.

According to THOMAS-KUHN ([3]), the condition:

$$\sum_i f_i + f_0 = 1$$

must be satisfied.

For values $x > 1$ (that are the most important in the ionization processes) the imaginagy terms in the denominator of the summation (11) can be eliminated.

The f_i and x_i values for H and He that we have adopted for the calculation are those reported in the Landolt-Börnstein tables ([4]).

The $f(x)$ for H has bee calculated by SUGIURA ([5]) and is given by

$$f(x) = \frac{2^7}{3}\frac{1}{x^4}\frac{\exp\left[-4/\sqrt{x-1}\right)\,\mathrm{tg}^{-1}\sqrt{x-1}]}{1-\exp\left[-2\pi/\sqrt{x-1}\right]}. \tag{13}$$

However, owing to the complicated form of (13) we will use, from now on, the approximate formula given by HÖNL ([6]), *i.e.*,

$$f(x) = \frac{h}{x^3} - \frac{k}{x^4}, \tag{14}$$

with h and k-values reported in Table I.

([3]) W. KUHN: *Zeits. Phys.*, **33**, 408 (1925); W. THOMAS: *Naturwiss.*, **13**, 627 (1925).
([4]) LANDOLT-BÖRNSTEIN: 6 Auflage, I Band, 1 Teil, pp. 260-262.
([5]) M. Y. SUGIURA: *Journ. Phys.*, **8**, 113 (1927).
([6]) H. HÖNL: *Zeits. Phys.*, **44**, 524 (1933).

TABLE I.

	h	k	f_0	I	$\tilde{I}/I$
H	1.0428	0.2607	0.4345	13.5	1.0880
H_2	1.0428	0.2607	0.4345	15.5	1.0000
He	2.6600	1.7670	0.7750	24.4	1.4090

(13) and (14) are practically identical up to $x = 6$. For greater values of x however, $f(x)$ rapidly vanishes and therefore the use of (15) instead of (14) cannot change the final results appreciably.

$f(x)$ for He has been theoretically calculated by VINTI and WHEELER (7) who also give rather complicated expressions. Therefore for simplicity, we will assume that the $f(x)$ for He can be also expressed by (14). The constant's h and k which appear in Table I have been obtained by fitting in Vinti's curve $f(x)$ (for the values of x near the absorption limit) and its integral value $f_0 = 0.775$. The validity of the theoretical $f(x)$ adopted in this work cannot be ascertained considering that, up to now, no accurate experimental tests on the behaviour of $f(x)$ near threshold are known (8).

With (14), the integral in the right hand side of (11) can be performed analytically and one finds the approximate formula (*)

$$
(15)\qquad \begin{cases} \operatorname{Re}\varepsilon(x) \simeq 1 + C\dfrac{n}{I^2}\left[\displaystyle\sum_i \frac{f_i}{x_i^2 - x^2} - \frac{f_0}{x^2} - \frac{k}{x^4}\log\left[(1-x^2)^2 + g^2x^2\right] + \right. \\ \qquad\qquad \left. + \dfrac{h}{x^4}\left(1 - \dfrac{1}{4x}\log\dfrac{(1 - x^2 - \frac{1}{4}g^2)^2 + g^2}{\left[(1-x)^2 + \frac{1}{4}g^2\right]^2}\right)\right], \\ \operatorname{Im}\varepsilon(x) \simeq \dfrac{\pi}{2} C\dfrac{n}{I^2}\dfrac{1}{x} f(x)\theta(x-1)\,, \end{cases}
$$

where

$$
\theta(x-1) = \begin{cases} 0 & \text{for } x < 1\,, \\ 1 & \text{for } x \geqslant 1\,. \end{cases}
$$

(7) J. P. VINTI: *Phys. Rev.*, **43**, 258 (1933); J. A. WHEELER: *Phys. Rev.*, **43**, 258 (1933).

(8) *Hand. d. Phys.*, 28, (Berlin, 1957), p. 110.

(*) For the total energy loss where, as known, the equation $1 - \beta^2\varepsilon(-iy) = 0$ has to be solved, we have used the expression obtained from the exact integration of (11).

4. – Mean specific primary, secondary and tertiary ionization in H and He.

Primary ionization: The calculation of the specific primary mean ionization has been performed in (I).

The expression for H and He (one ionization limit) can be written in the form

$$\frac{\partial N_1(E)}{\partial z} = \frac{2\pi e^4 n}{m v^2 I}\left\{1 - \frac{I}{T} + \int_1^\infty \frac{f(x)\,\mathrm{d}x}{x}\left[\log\frac{2mv^2}{Ix^2\,|1-\beta^2\varepsilon^*(x)|} - \beta^2\,\mathrm{Re}\,\varepsilon(x)\right]\right\}, \tag{16}$$

where $\varepsilon(x)$ is given by (15) and T is the maximum energy transferable from the primary particles to an electron. For values of T of experimental interest (16) is practically independent from T.

Secondary ionization: The mean specific secondary ionization can be calculated from

$$\frac{\partial N_2(E)}{\partial z} = \int_0^\infty N_1^e(\varepsilon)\,\frac{\partial n_1(\varepsilon, E)}{\partial z}\,\mathrm{d}\varepsilon\,, \tag{17}$$

with

$$N_1^e(\varepsilon) = -\int_I^\varepsilon \frac{\partial N_1^e(\varepsilon)}{\partial z}\left(\frac{\partial \varepsilon}{\partial z}\right)^{-1}\mathrm{d}\varepsilon\,. \tag{18}$$

As the energy of the electrons coming out from the primary ionization is generally low, $N_1^e(\varepsilon)$ can be calculated by introducing in (18) the $\partial N_1^e(\varepsilon)/\partial z$ deduced from (16) in the limit $\beta^2 = 0$ and $E = mv^2/2$. One then obtains, putting the maximum transferable energy in electron-electron collisions $T = E/2$

$$\frac{\partial N_1^e(E)}{\partial z} = \frac{\pi e^4 n}{E I}\left[\left(1 - \frac{2I}{E}\right)\theta(E - 2I) + H_1 \log\frac{4E}{I} - K_1\right], \tag{19}$$

where

$$\begin{cases} H_1 = \displaystyle\int_1^\infty \frac{f(x)\,\mathrm{d}x}{x} = \frac{h}{3} - \frac{k}{4}, \\ K_1 = 2\displaystyle\int_1^\infty \frac{f(x)\log x\,\mathrm{d}x}{x}\,\mathrm{d}x = 2\left(\frac{h}{9} - \frac{k}{16}\right). \end{cases} \tag{20}$$

The values of H_1 and K_1 for hydrogen and He are reported in Table III.

Furthermore (7) of (I) in the limit $\beta \simeq 0$ and $E = mv^2/2$ is given by

$$-\frac{\partial E}{\partial z} = \frac{2\pi e^4 n}{E}\left[\log\frac{E}{\tilde{I}}\,\theta(E-\tilde{I}) + \frac{1}{2}\log 2\right], \tag{21}$$

with

$$\tilde{I} = I\,x_1^{f_1}\cdot x_2^{f_2}\ldots \exp\left[\frac{h}{4} - \frac{k}{9}\right]. \tag{22}$$

In Table I one finds the values of $\tilde{I}/I$ for H, H_2 and He.

The factor $\theta(E-2I)$ in (19) takes into account that the maximum transferable energy from the primary electron to the ionized one cannot be less than I. The factor $\theta(E-\tilde{I})$ has been introduced in (21) because the term $\log E/\tilde{I}$ loses its physical meaning when $E < \tilde{I}$. (21) coincides with the Bethe formula [9].

As far as the expression (19) is concerned, it fits the experimental data better than the Born approximation (with a deviator lower than 25%) in the region $2I \leqslant E \leqslant 100$ eV while for $E > 100$ eV it practically coincides with this one [10]. For $E < 2I$ however, (19) deviates very much from the Born approximation (which obviously is also incorrect at this energy) but considering that only the ratio between specific ionization and total energy loss is important for our purpose, and that both these expressions are calculated in the same approximation, it is to be hoped that at low energies the errors will compensate. The final results, in fact, convalidate this hypothesis.

With (19) and (21) $N_1^e(E)$ then becomes

$$N_1^e(E) = \frac{1}{2I}\int_I^E \frac{(1-2I/E)\theta(E-2I) + H_1\log 4E/I - K_1}{\log(E/\tilde{I})\,\theta(E-\tilde{I}) + \frac{1}{2}\log 2}\,\mathrm{d}E\,, \tag{23}$$

which can be integrated exactly, by elementary methods.

Taking for $\partial n_1(\varepsilon, E)/\partial z$ the integrand of (16), the mean specific secondary ionization is

$$\frac{\partial N_2(E)}{\partial Z} = \frac{2\pi e^4 n}{mv^2 I}\cdot \left\{\int_2^{T/I} N_1^e(E-I)\,\frac{\mathrm{d}x}{x^2} + \int_2^{\infty}\frac{N_1^e(E-I)\,f(x)}{x}\left[\log\frac{2mv^2}{Ix^2\,|1-\beta^2\varepsilon^*(x)|} - \beta^2\right]\mathrm{d}x\right\}, \tag{24}$$

where $N_1^e(E)$ is given by (23).

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[9] H. Bethe: *Zeits. Phys.*, **76**, 293 (1932).

[10] H. S. W. Massey and E. H. S. Burhop: *Electronic and Ionic Impact Phenomena* (1952), p. 151.

Tertiary specific ionization: From (7), for $n = 2$ one obtains:

$$N_2^e(E) = \int_{2I}^{E} \left(\frac{\partial N_2^e(E)}{\partial z}\right) \cdot \left(\frac{\partial E}{\partial z}\right)^{-1} \mathrm{d}E \,, \tag{25}$$

which can be calculated taking for $\partial N_2^e(E)/\partial z$ the (24) in the limit $\beta = 0$ and $E = mv^2/2$ and for $\partial E/\partial z$ the expression (21).

By means of (25), the tertiary specific ionization $\partial N_3(E)/\partial z$ can be deduced from (5) with $n - 2$ and where in place of $\partial n_1(\varepsilon, E)/\partial z$ one can again put the integrand of (16).

In the same way, the specific ionization of the n-th generation can be obtained.

Numerical results: The exact calculation of the mean specific total, primary, secondary and tertiary ionization (other generations being neglegible) together with the total energy loss in H and He has been performed by a IBM 650 computing machine. The curves together with the experimental data are reported in Figs. 1-8.

In Fig. 1 one can see the results for H at 0 °C, 1 atm. and $T = 56$ keV

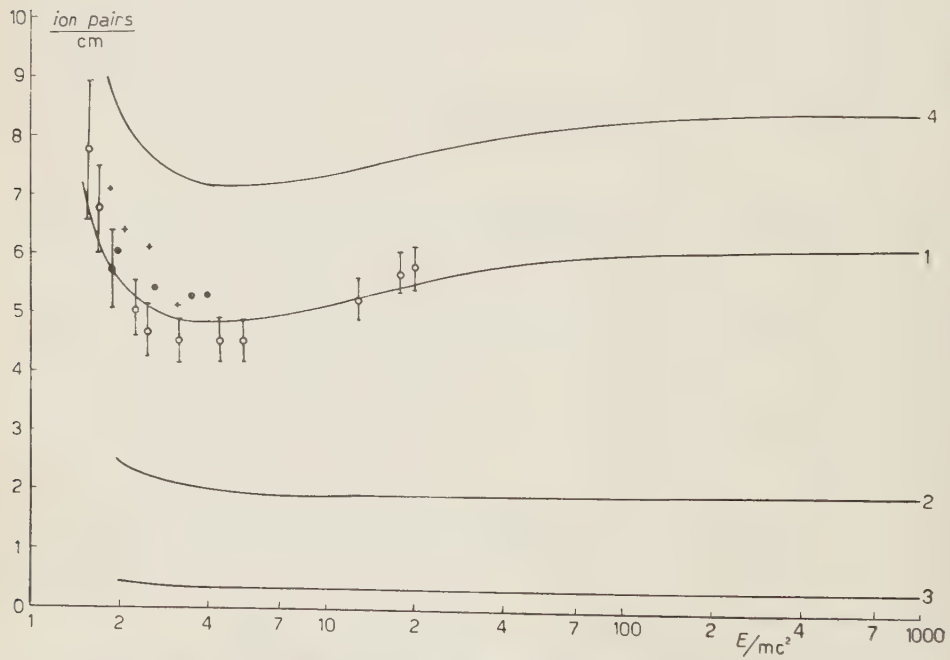

Fig. 1. – H, 1 atm, 0 °C, $T = 56$ keV. 1) primary ionization; 2) secondary ionization; 3) tertiary ionization; 4) total ionization; ● McClure's data; + Heisenberg's data; ○ Hereford's data (primary ionization only, see ref. [11]).

for $I = 13.5$ eV. The experimental data of various authors ([11]) refer to the primary ionization only. Fig. 2 shows the data for H at 23 °C, 1 atm and $T = 56$ keV together with the data of BARBER and FRETTER ([12]). The higher

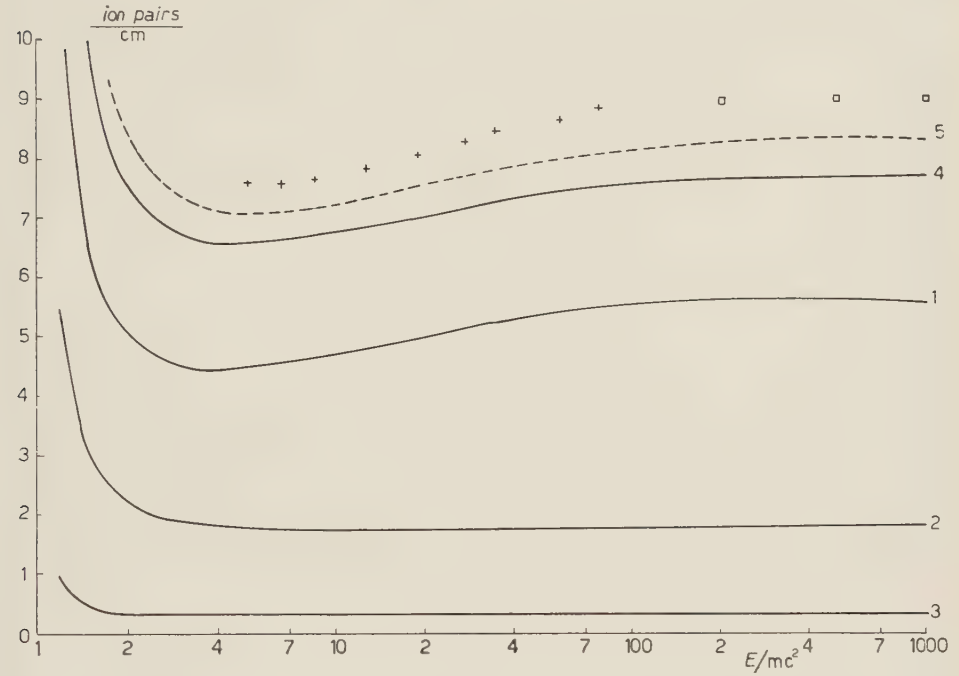

Fig. 2. – H, 1 atm, 23 °C, $T = 56$ keV. 1) primary ionization; 2) secondary ionization; 3) tertiary ionization; 4) total ionization; 5) total ionization calculated adding (39) to primary one $w = 35.3$ eV, $I = 13.5$ eV; + Barber's data (total ionization, see ref. ([12])); □ Fretter's data.

dotted curve represents the total specific ionization calculated by adding the expression (39) (see Section 6) to the primary one (for. (31)). It is seen that the absolute value of the primary specific ionization is in satisfactory agreement with the experimental results for H. The theoretical total specific ionization, however, is less than the experimental values. This may be due to the ignored contribution of generations higher than the third one and to the contribution of other gases present in the experimental apparatus (*).

([11]) G. M. MCCLURE: *Phys. Rev.*, **90**, 796 (1953; W. HEISENBERG: *Kosmische Strahlung*, p. 492; F. L. HEREFORD: *Phys. Rev.*, **74**, 574 (1958).

([12]) W. C. BARBER: *Phys. Rev.*, **103**, 1281 (1956); W. B. FRETTER: these data have been kindly communicated to us by Dr. T. L. AGGSON (1959). They have been normalized at Barber's minimum.

(*) These contributions will be discussed in a work to be published.

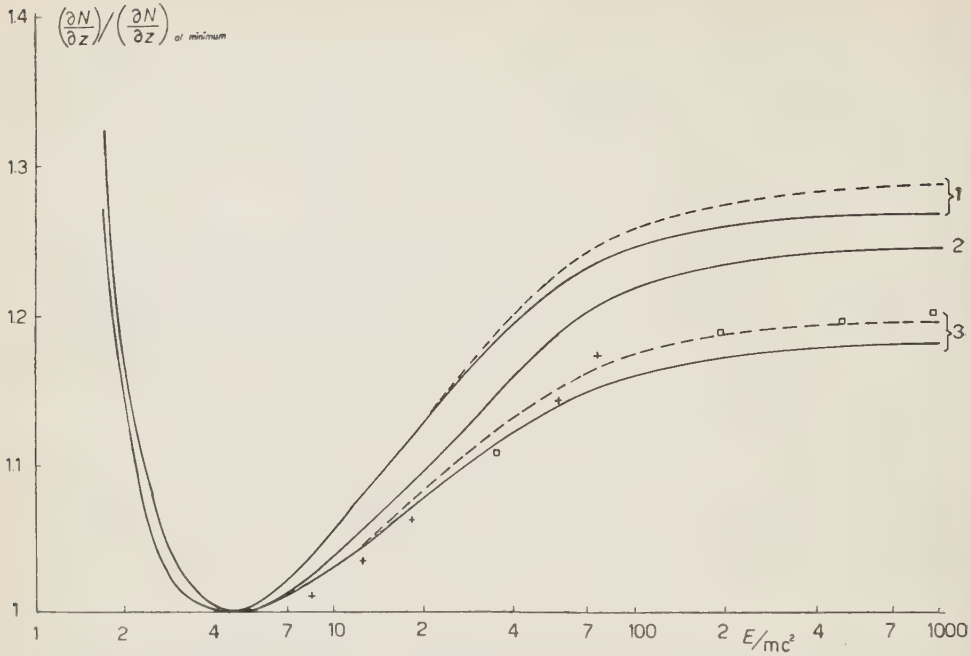

Fig. 3. – Relativistic increase. H, 1 atm, 23 °C, $T=56$ keV. 1) primary ionization (full line: $I=13.5$ eV; dotted line: $I=15.5$ eV); 2) total energy loss; 3) total ionization; + Barber's data (ref. [12]); □ Fretter's data.

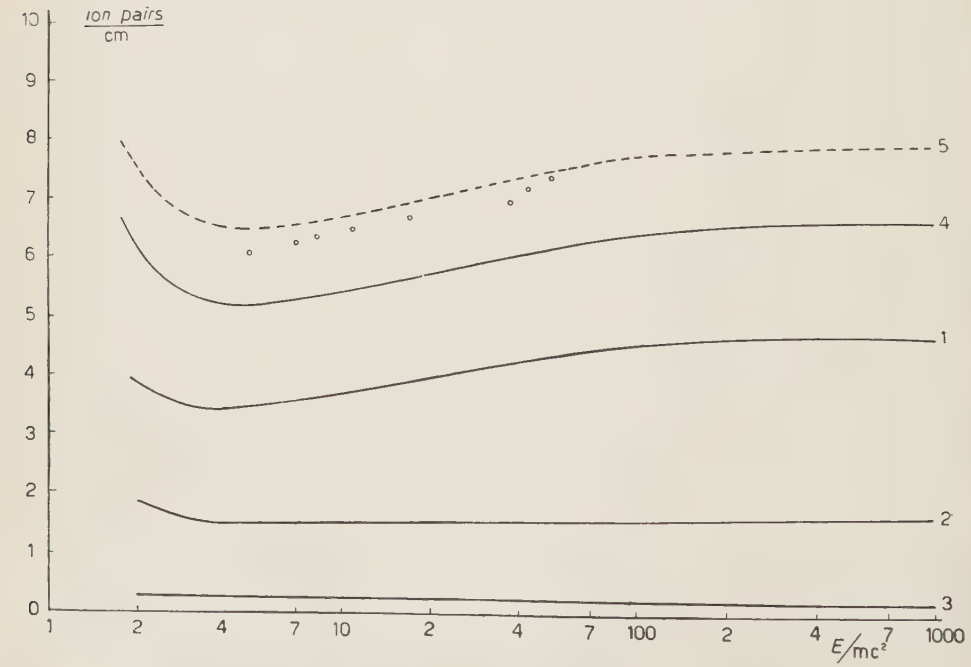

Fig. 4. – He, 1 atm, 23 °C, $T=51$ keV. 1) primary ionization; 2) secondary ionization; 3) tertiary ionization; 4) total ionization; 5) total ionization calculated adding (39) to the primary one ($w=29.9$ eV, $-I=24.4$ eV); ○ Barber's data (see ref. [12]).

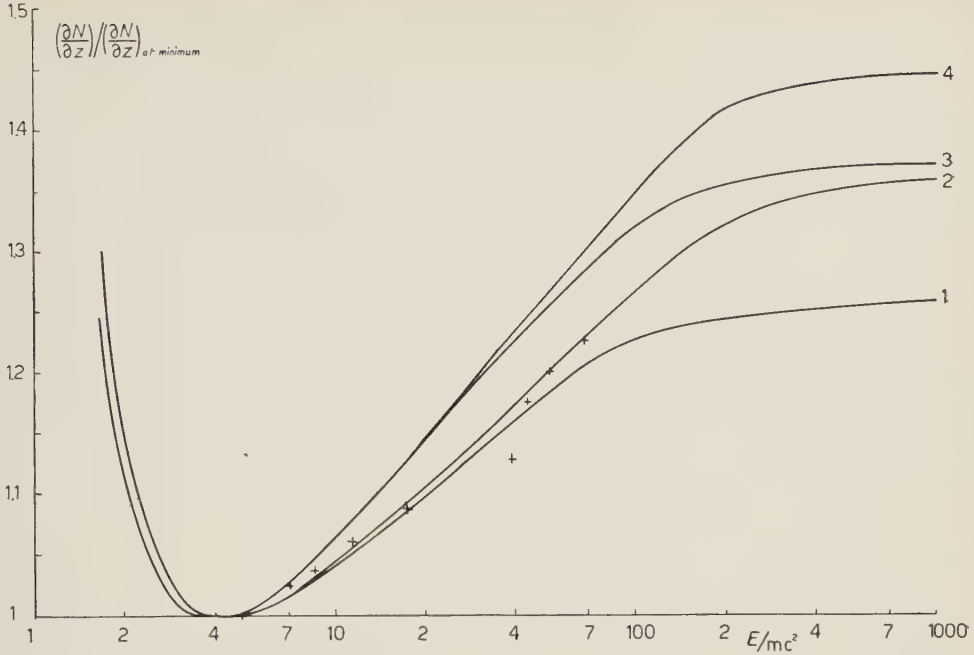

Fig. 5. – Relativistic increase. He, 1 atm, 23 °C. 1) total ionization ($T=51$ keV); 2) total energy loss ($T=51$ keV); 3) primary ionization; 4) total energy loss ($T=1$ keV); + Barber's data (see ref. [12]).

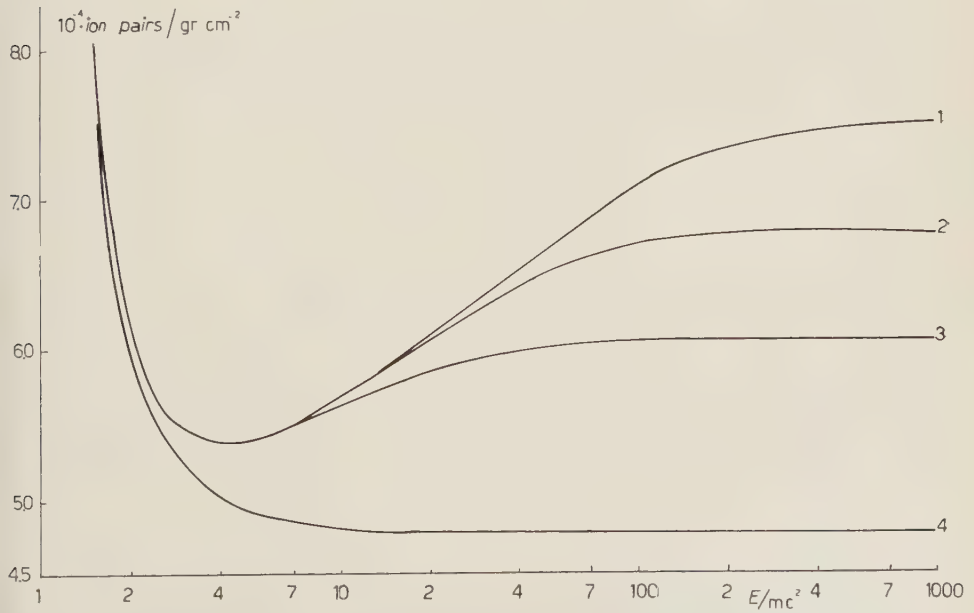

Fig. 6. – Primary ionization. H, 0 °C. 1) 0.1 atm relativistic increase = 40%; 2) 1 atm relativistic increase = 26%; 3) 10 atm relativistic. increase = 12%; 4) liquid no relativistic increase.

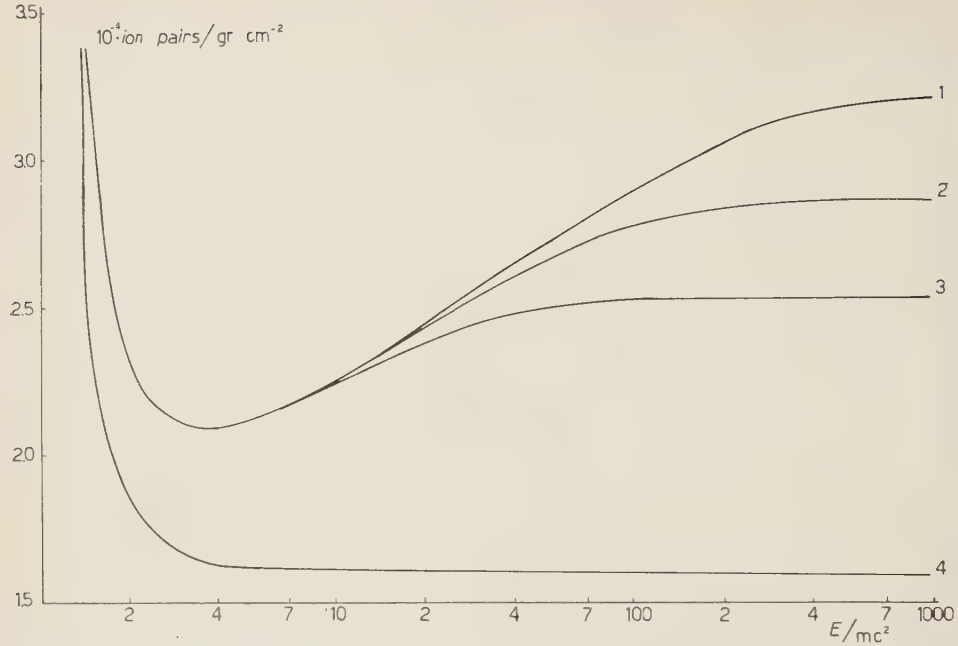

Fig. 7. – Primary ionization. He, 0 °C. 1) 0.1 atm relativistic increase = 53%; 2) 1 atm relativistic increase = 36%; 3) 10 atm relativistic increase = 21%; 4) liquid no relativistic increase.

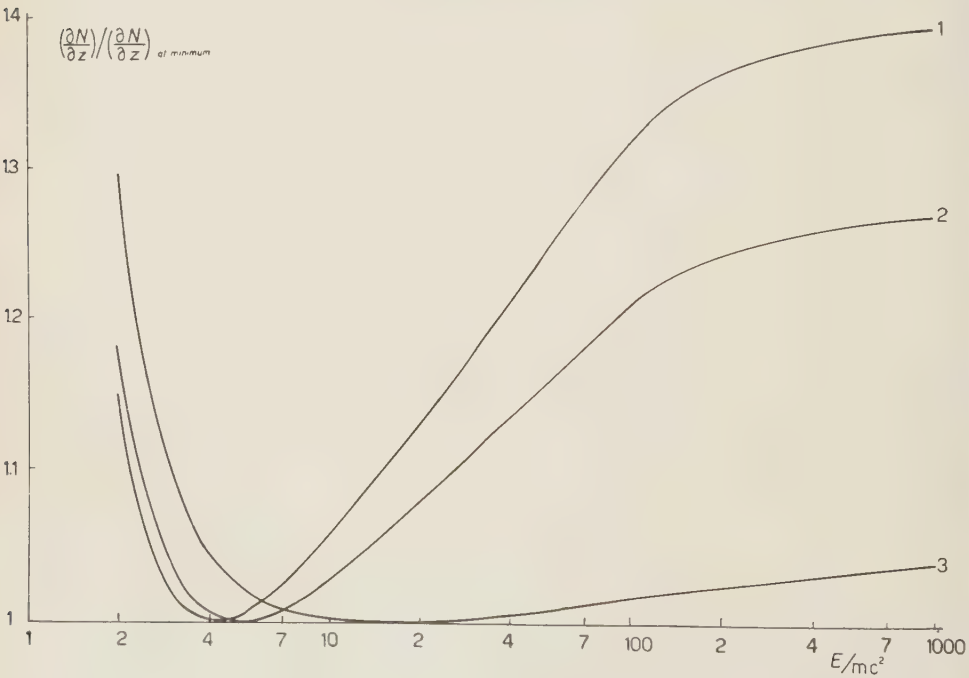

Fig. 8. – Relativistic increase. H, 0 °C, 0.1 atm, $T = 56$ keV. 1) primary ionization; 2) total ionization; 3) secondary ionization.

Further, it is seen that, at the density considered, already in the secondary ionization relativistic increase is not present. This explains why the relativistic increase is less for the total than for the primary ionization as shown in Fig. 3, where we have reported the behaviour of the relativistic increase of the total energy loss [(49)-(I)], total specific ionization, primary specific ionization and Barber and Fretters' experimental data for H at 1 atm 23 °C. It would be interesting to obtain data on the relativistic increase for primary ionization in H which, according to the results reported in Fig. 3. should be about 1.28 as compared to the 1.19 for total ionization. One further effect of the dilution of the primary ionization due to higher generations is the shifting of the position of the minimum towards the higher energies. This effect would become more important as the density of the medium decreases *i.e.* as the relativistic increase of the secondary ionization becomes appreciable. This can be seen in Fig. 8 where the relativistic increase of the primary, secondary and total specific ionization for H at 0.1 atm and 0 °C has been reported.

In Figs. 4 and 5 we have reported the results for He at 1 with $T = 51$ keV. Helium data show a behaviour similar to the H ones. Here the agreement with the experimental results is something better both for the absolute value and for the relativistic increase. This is conforting since He does not present the difficulty of the molecular structure as H_2 does, thus the theoretical data are more trustworthy.

Figs. 6 and 7 show the behaviour of the primary ionization for g/cm² for H and He respectively at various densities of the media. One can then see that liquid H and He should not present relativistic increase.

5. – Simplified formulae.

Because of the complicated expressions of $\varepsilon(x)$, (16) cannot be integrated analytically. This fact, apart from the disadvantage of not having simple formulae to be adapted to the particular experimental situation, also prevents the direct inspection of the influence of the different physical parameters on the phenomenon. We have thus tried to extract from the theory, simplified expressions which reproduce, with reasonable precision, the results of the theory.

Primary ionization: We start from the consideration that (16) for $E - mc^2 \to 0$ ($\beta \ll 1$) gives the same results as the theory without density effect which can be obtained putting $\varepsilon(x) = 1$. For $E \to \infty$, (16) goes to the limit obtained replacing throughout $\beta = 1$. We establish, thus the condition

that the required simplified formula for $\partial N_1(E)/\partial z$, must be identical with the exact one at these two limits. For this aim, let us consider the integral

$$(26)\quad J_\infty = \lim_{E\to\infty} \frac{2\pi e^4 n}{mv^2 I} \int_1^\infty \frac{f(x)}{x} \log|1-\beta^2\varepsilon^*(x)|^2\,\mathrm{d}x = \frac{2\pi e^4 n}{mc^2 I} \int_1^\infty \frac{f(x)}{x} \log|1-\varepsilon^*(x)|^2\,\mathrm{d}x\,,$$

where

$$|1-\varepsilon^*(x)|^2 = C^2 \frac{n^2}{I^4} Q^2(x)\,,$$

C is given by (12) and

$$(27)\quad Q^2(x) =$$

$$= \left|\sum_i \frac{f_i}{x_i^2 - x^2} - \frac{f_0}{x^2} - \frac{h}{4x^4}\log[(1-x^2)^2+g^2x^2] + \frac{k}{x^4}\left(1 - \frac{1}{4x}\log\frac{(1-x^2-g^2/4)^2+g^2}{[(1-x)^2+\frac{1}{4}g^2]^2}\right)\right|^2 + \frac{\pi^2}{4}\frac{1}{x^2} f^2(x)\theta(x-1)\,.$$

Evaluating (26) with the mean value method and inserting it in (17), we obtain

$$(28)\quad \lim_{E\to\infty} \frac{\partial N_1(E)}{\partial z} = \frac{2\pi e^4 n}{mc^2 I}\left|1 - \frac{I}{T} + H_1\left(\log\frac{2mc^2 I}{nC\bar{Q}_1} - 1\right) - K_1\right|,$$

where H_1 and K_1 are defined by (22) and

$$(29)\quad \log\bar{Q}_1 = \frac{\int_1^\infty f(x)/x\,\log Q(x)\,\mathrm{d}x}{\int_1^\infty f(x)/x\,\mathrm{d}x} = \frac{1}{H_1}\int_1^\infty \frac{f(x)}{x}\log Q(x)\,\mathrm{d}x\,.$$

We see thus that $\bar{Q}_1$ (as well as H_1 and K_1) are independent of the density for a given medium and can be determined by its dielectric characteristics. The calculated values of $\bar{Q}_1$ for H and He are listed in Table II.

TABLE II.

	H_1	H_2	K_1	K_2	$\bar{Q}_1$	$\bar{Q}_2$	p	C_1	C_2
H	0.2824	0.0407	0.1992	0.1257	0.637	0.0503	0.264	0.096	0.31
H_2	0.2824	0.0407	0.1992	0.1257	0.637	0.403	0.239	−0.0299	0.31
He	0.4500	0.0926	0.3728	0.2913	0.845	0.0817	0.429	−0.143	0.31

In the limit $\varepsilon(x) = 1$ (valid for low energy), and with $f(x)$ given by (14), (16) can be integrated exactly and one obtains (*)

$$(30) \qquad \left.\frac{\partial N_1(E)}{\partial z}\right|_{\varepsilon(x)=1} = \frac{2\pi e^4 n}{mv^2 I}\left[1 - \frac{I}{T} + H_1\left(\log\frac{2mc^2\beta^2}{I(1-\beta^2)} - \beta^2\right) - K_1\right].$$

As we have said, our purpose is to write an approximate formula which exactly reproduces (28) for $\beta = 1$ and (30) for $\beta^2 \ll 1$ where one can assume $\varepsilon(x) = 1$. One can easily see that both these conditions are satisfied by the expression

$$(31) \qquad \frac{\partial N_1(E)}{\partial z} = \frac{2\pi e^4 n}{mv^2 I}\left[1 - \frac{I}{T} + H_1\left(\log\frac{2mc^2\beta^2}{I[(1-\beta^2)^2 + \beta^4 n^2 (C^2/I^4)\bar{Q}_1^2]^{\frac{1}{2}}} - \beta^2\right) - K_1\right].$$

We have then compared the values obtained from the numerical integration of (16) with those obtained with (31) for H at 1 and 10 atm. The two results practically coincide inside the interval of $\gamma = E/mc^2$ between 1.5 and ∞.

The maximum difference is presented at γ given by (32) and is less than 2 % in H and He at 10 atm (the approximate formula exceeds the exact one). The deviations decrease as the density decreases.

Thus (31) may serve both for comparison with the experimental results and for discussion of the physical implications of the theory.

The terms proportional to n^2 inside the logarithm gives the saturation effects which set in at such an energy that

$$1 - \beta^2 \simeq \beta^2 n \frac{C}{I^2}\bar{Q}_1,$$

that is

$$(32) \qquad (\beta\gamma)^2 \simeq \frac{I^2}{nC\bar{Q}_1}.$$

Apart from the linear dependence on the reciprocal of density $\varrho = (A/Z)n$, one observes a dependence on the square of the ionization potential. This dependence is confirmed by the results reported in Figs. 3 and 6 where one can see that the density effect is stronger in H ($I = 13.5$) than in H_2 ($I = 15.5$) and in He ($I = 24.4$).

Secondary and tertiary ionization: In order to give an approximate formula for the total specific ionization one must add to (31) the part referring

(*) For H, (30) coincides practically with the well known Bethe formula deduced by a different method. A generalization of (30) valid also for other elements will be discussed in a subsequent work.

to the secondary, tertiary etc., specific ionization. Because in this case also no exact integrations can be performed, we have tried to approximate the exact expressions.

For the secondary ionization we start by assuming the following approximate formula for $N_1(E)$

$$N_1^e(E) = p\left(\frac{E}{I} - 1\right), \tag{33}$$

where the constant p for H and He are determined in order to approximate the exact expression (23) and are listed in Table II.

Substituting (32) in (24) and performing the integrations we obtain

$$\frac{\partial N_2(E)}{\partial z} = \frac{2\pi e^4 n}{mv^2 I}\, p\left[\log\frac{T}{2I} - 1 + \frac{2I}{T} + H_2\left(\log\frac{2mv^2}{I} - \beta^2\right) - K_2 - \int_2^\infty \frac{f(x)(x-2)}{x}\log|1-\beta^2\varepsilon^*(x)|\,\mathrm{d}x\right], \tag{34}$$

where

$$\left|\begin{aligned} H_2 &= \int_2^\infty \frac{f(x)(x-2)}{x}\,\mathrm{d}x = \frac{h}{24} - \frac{k}{96}, \\ K_2 &= 2\int_2^\infty \frac{f(x)(x-2)}{x}\log x\,\mathrm{d}x = \frac{h}{4}\left[\frac{1}{3}\log 2 + \frac{5}{18}\right] - \frac{k}{48}\left[\log 2 + \frac{7}{12}\right]. \end{aligned}\right. \tag{35}$$

As a second step, calculating the integral in (34) with the mean value method (as has been made for the primary ionization) we have

$$\frac{\partial N_2(E)}{\partial z} = \frac{2\pi e^4 n}{mv^2 I}\, p \left\{\log\frac{T}{2I} - 1 + \frac{2I}{T} + H_2\left[\log\frac{2mv^2}{I\left[(1-\beta^2)^2 + \beta^4 n^2 (C^2/I^4)\bar{Q}_2^2\right]^{\frac{1}{2}}} - \beta^2\right] - K^2\right\}, \tag{35}$$

where the values of

$$\log\bar{Q}_2 = \frac{\int_2^\infty (f(x)/x)\,(x-2)\log Q(x)\,\mathrm{d}x}{\int_2^\infty (f(x)(x-2)/x)\,\mathrm{d}x},$$

are reported in Table II.

Because of the very small values of H_2 and K_2 (see Table II) the first three terms of (35) (close collision terms) generally prevail on the other ones (col-

lision terms). For example for H at standard conditions and with $T = 56$ keV the close collision term in (35) is about $(8 \div 10)$ times greater than the distant collision one. This is the obvious physical consequence of the fact that secondary ions come out prevailingly from the more energetic collisions which are close collisions. Furthermore this explains the stronger dependence of (35) on the maximum transferable energy T as compared to that of the primary ionization and the very poor relativistic increase of the secondary ionization.

For the tertiary ionization, taking for $\partial n_1(\varepsilon, E)/\partial z$ the simple expression $(2\pi e^4 n/mv^2 I)\cdot(1/x^2)$ (distant collisions for tertiary and higher generations ionizations are negligible), from (5) with $n = 2$ the numerical calculation gives

$$\frac{\partial N_3(E)}{\partial z} = \frac{2\pi e^4 n}{mv^2 I}\, p[C_1 + C_2 \log T]\,, \tag{37}$$

where T is measured in keV.

Adding together (31), (36) and (37) one obtains the required approximate formula for the total specific ionization.

Another simplified approach for calculating total specific ionization: The total ionization can be obtained (see Section **2**) adding to (31) the

$$\sum_{n=1}^{n_{\max}} \frac{\partial N_{n+1}(E)}{\partial z} = \int_0^\infty N_T^e(\varepsilon)\, \frac{\partial n_1(\varepsilon, E)}{\partial z}\, \mathrm{d}\varepsilon = \tag{38}$$

$$\frac{2\pi e^4 n}{mv^2 I}\left\{\int_2^{T/I} N_T^e(x-1)\,\frac{\mathrm{d}x}{x^2} + \int_2^\infty \frac{N_T^e(x-1)\,f(x)}{x}\left[\log \frac{2mv^2}{Ix^2\,|1-\beta^2\varepsilon^*(x)|} - \beta^2\right]\right\},$$

where $N_T^e(E)$ is given by (10). Unfortunately the lack of knowledge about $w(E)$ especially for low energies does not allow us the exact calculation of (38). Let us suppose for a moment that $w(E)$ is constant for all energies. Then the integral (10) can be exactly performed

$$N_T^e(x) = \frac{I}{w}\,(x-1)\,.$$

By inserting this in (38) and performing the integrations we find

$$\sum_{n=1}^{n_{\max}} \frac{\partial N_{n+1}}{\partial z} = \frac{2\pi e^4 n}{mv^2 w}\cdot \tag{39}$$

$$\cdot\left\{\log\frac{T}{2I} - 1 + \frac{2I}{T} + H_2\left[\log\frac{2mc^2\beta^2}{I[(1-\beta^2)^2 + \beta^4 n^2 (C^2/I^4)\bar{Q}_2^2]^{\frac{1}{2}}} - \beta^2\right] - K_2\right\},$$

where the integral of (38) containing the factor $\log|1-\beta^2\varepsilon^*(x)|$ has been calculated with the approximated method used in calculating (35).

Experimental results of various authors ([13]) have pointed out that $w(E)$ is practically constant for high energies and increases as the energy decreases. Thus, the constancy of $w(E)$ is incorrect in this case. However, the upper limit of the total specific ionization can be obtained adding (31) to (39) where w is the experimental value at high energy. This upper limit has been reported in Figs. 2 and 4 (higher dotted curve) for H and He at 1 atm putting $w = 35.5$ eV and 29.9 eV respectively. We have obtained the result that although the theoretical value for He is higher than the experimental data, for H it is lower. This could be due to the fact that the primary specific ionization (31) for He is more exact than that for H owing to the influence of the molecular structure of H_2 in the ionization process.

6. – Conclusion.

The calculations, developed in the previous sections, show that the mean specific ionization, both primary and total, can be computed with reasonable accuracy for H and He and for practically all energies of the ionizing particle, including those for which the density effect is dominant. Further the final formula can be put in simplified form where the dependence on the physical parameters appears evident.

One of the principal results of these calculations is that they have shown the different relativistic behaviour of primary ionization, total ionization and total energy loss. In fact while these two depend strongly on the maximum transferable energy T, an empirical parameter dependent on the particular experiment, (see curves 1, 2 and 4 in Fig. 5), primary specific ionization is practically independent on T. Further the relativistic increase of primary ionization is larger than that of total ionization, due to the fact that ionization of higher generations is mainly produced by non relativistic electrons produced in knock-on collisions and thus does not contribute to relativistic increase. These results, together with the consideration that the independence from T means no straggling, might recommend to base experiments on relativistic increase over effects depending principally on primary specific ionization.

We think that this different relativistic behaviour of primary and total ionization might be also an explanation for the disagreeing experimental results found by different authors who do not clearly separate effects due to primary and total ionization.

([13]) E. SEGRÈ: *Experimental Nuclear Physics*, vol. **1** (1952), p. 232.

One further source of variation of the experimental results could be also the presence of other gases in the ionized medium. In order to treat this problem we have to generalize the present theory for complex atoms and for mixtures of elements, generalizations which will be discussed in a subsequent paper.

RIASSUNTO

Si dà un metodo per il calcolo del valore assoluto della ionizzazione specifica primaria, secondaria e totale generata da una particella ionizzante che attraversa un dato mezzo. Il metodo tiene conto dell'effetto di densità. Si danno risultati numerici per l'idrogeno e per l'elio. La ionizzazione primaria presenta un aumento relativistico maggiore che non la ionizzazione totale nella quale si fa sentire la diluizione dovuta alla ionizzazione secondaria che presenta scarso aumento relativistico. La ionizzazione primaria è inoltre indipendente dalla massima energia trasferibile negli urti stretti che influenza invece notevolmente la ionizzazione delle generazioni più alte. Si danno formule semplificate nelle quali appare esplicitamente la dipendenza del fenomeno in istudio dalle caratteristiche fisiche del mezzo, adatta al confronto della teoria con i risultati sperimentali.

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

A Remark on the Theory of Fermions with Higher Order Derivatives.

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(ricevuto il 1° Agosto 1960)

Summary. — The problem of negative norms arising in a multifermion theory derived from a single Lagrangian containing derivatives of order higher than the first may be removed by the adoption of a suitable indefinite metric. Interaction S-matrix elements, calculated with the metric operator, are shown to be equivalent to those of the conventional theory.

1. – Introduction.

Some years ago, Pais and Uhlenbeck (1) considered the possibility of using higher-order and non-local Lagrangians for the description of fundamental interactions, in an attempt to construct a theory containing only finite (regularized) S-matrix elements. As is well known, all such attempts have been unsuccessful. Although the results of the present investigation do not change this situation, there are several differences of method which are perhaps of sufficient interest in themselves to warrant a separate discussion. In particular, we would like to re-examine the question of a set of multifermion fields whose properties stem from a Lagrangian containing derivatives of order higher than the first, for the purpose of demonstrating the complete physical consistency of such a scheme. Rather than postulate anticommutation rules

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(1) A. Pais and G. E. Uhlenbeck: *Phys. Rev.*, **79**, 145 (1950). Other pertinent references are quoted in this paper.

in the manner of reference [1], these may be obtained from a purely canonical procedure (application of the Action Principle); but one immediately finds that certain of the anticommutators so obtained are negative, which point suggests the introduction of a suitable indefinite metric in order to insure positive probabilities. Once this artifice has been adopted, the calculation of any interaction S-matrix element leads to results identical to those obtained from a conventional theory of distinct fermions, where each field is, at the outset, described by its own first-order Lagrangian.

This treatment suggests a more lenient attitude than is customary towards the concept of an indefinite metric. Starting from the usual first-order fermion Lagrangian, application of the Action Principle together with the conventional probability interpretation leads to an anticommutation algebra which permits, and is consistent with, a positive metric in the Hilbert space of the corresponding operators; and a theory of several fermions is then constructed as a superposition of several such Lagrangians and Hilbert spaces. From a higher-order Lagrangian, but using only the *same* first principles, one obtains a theory in which the metric of certain fields must be indefinite. Since the physical predictions of either theory are indistinguishable, there is no *a priori* reason for not taking the indefinite metric seriously [2].

2. – The multifermion formulation.

Consider the n-th order Lagrangian [3]

$$L = -\bar{\psi}(\delta + m_2) \dots (\delta + m_n)\psi \equiv -\bar{\psi}[\Pi(\delta + m)]\,\psi\,, \tag{1}$$

where none of the mass parameters m_i are equal; for simplicity the Lagrangian is here written in unsymmetrized form. Application of the Action Principle yields the field equations

$$[\Pi(\delta + m)]\psi = \bar{\psi}[\Pi(\underset{\leftarrow}{\delta} - m)] = 0\,, \tag{2}$$

together with the generators of infinitesimal variations of ψ,

$$G = \int \mathrm{d}\sigma_\mu \{-\bar{\psi}\gamma_\mu(\delta + m_2) \dots (\delta + m_n)\,\delta\psi + \bar{\psi}(\underset{\leftarrow}{\delta} - m_1)\gamma_\mu(\delta + m_3) \dots (\delta + m_n)\,\delta\psi + \dots + \\ + (-1)^n \bar{\psi}(\underset{\leftarrow}{\delta} - m_1) \dots (\underset{\leftarrow}{\delta} - m_{n-1})\gamma_\mu\,\delta\psi\}\,.$$

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[2] This statement is at variance with a recent remark of J. SCHWINGER: *Phys. Rev.*, **115**, 721 (1959).

[3] We use the notation:

$$P_\mu = (\boldsymbol{p}, iP_0)\,, \qquad \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}\,, \qquad \gamma_\mu^+ = \gamma_\mu\,, \qquad \delta = \gamma_\mu\partial_\mu\,, \qquad \bar{\psi}\underset{\leftarrow}{\delta} = \partial_\mu\bar{\psi}\gamma_\mu\,.$$

For simplicity, individual spinor indices are suppressed. Since we are dealing with an n-th order Lagrangian, we must specify an independent variations; and this is accomplished by postulating the relations

$$\{\psi, \delta\psi'\} = \{(\eth + m_n)\psi, (\eth' + m_n)\,\delta\psi'\} =$$

$$= \{(\eth + m_n)(\eth + m_{n-1})\psi, (\eth' + m_n)(\eth' + m_{n-1})\,\delta\psi'\} = \ldots = 0\,,$$

where $\psi = \psi(\boldsymbol{x}, x_0)$ and $\psi' = \psi(\boldsymbol{x}', x_0)$. These statements are essentially equivalent to the assumptions

$$\{\partial_0^l\psi,\ \partial_0^m\,\delta\psi'\} = 0\,, \qquad\qquad n \geqslant l,\ m \geqslant 0.$$

There then follow the equal-time anticommutation rules

$$(3)\qquad \begin{cases} \{\psi, \overline{\psi}\,(\underset{\leftarrow}{\eth'} - m_1) \ldots (\underset{\leftarrow}{\eth'} - m_{n-1})\} = (-1)^{n-1}\gamma_4\,\delta(\boldsymbol{x} - \boldsymbol{x}')\,, \\ \{(\eth + m_n)\psi, \overline{\psi}'(\underset{\leftarrow}{\eth'} - m_1) \ldots (\underset{\leftarrow}{\eth'} - m_{n-2})\} = (-1)^{n-2}\gamma_4\,\delta(\boldsymbol{x} - \boldsymbol{x}')\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \{(\eth + m_n) \ldots (\eth + m_2)\,\overline{\psi},\ \psi'\} = \gamma_4\,\delta(\boldsymbol{x} - \boldsymbol{x}'). \end{cases}$$

Denoting the c-number anticommutator $i\{\psi(x), \overline{\psi}(x')\}$ by $S(x - x')$, we then have $[\Pi(\eth + m)]S = 0$, and the relations of eq. (3) may be viewed as n initial conditions which uniquely determine the solution of this n-th order hyperbolic equation (4). To construct this quantity it is convenient to first examine the solutions of eq. (2). We expect the latter to be given in terms of solutions to n first-order Dirac equations, $(\eth + m_i)\psi_i = 0$, which property may be concisely expressed in terms of projection operators

$$\psi = \sum_{i=1}^{n} \psi_i\,, \qquad \text{if} \quad \psi_i \equiv P_i\psi\,,$$

where $\sum_{i=1}^{n} P_i = 1$, and $P_iP_j\,\psi = \delta_{ij}P_i\psi$.

If we write

$$P_i = c_i(\eth + m_1)(\eth + m_2) \ldots \bigwedge_i \ldots (\eth + m_n)\,,$$

(4) These conditions are just another way of stating the requirements: $\partial_0^j S\,|_{x_0 = x_0'} = 0$ for $n - 1 \geqslant j \geqslant 0$, $\partial_0^{n-1} S\,|_{x_0' = x_0} = (i\gamma_4)^n\,\delta(\boldsymbol{x} - \boldsymbol{x}')$, which follow from eq. (3) upon using the invariance of the Lagrangian under the interchange of any of the m_i. Alternately, they may be obtained directly by re-writing $L(\eth, \eth^2, \eth^3, \ldots)$ as $L(\eth, \Box, \eth\Box, \ldots)$ before performing the variations.

where c_i is a numerical constant, and the symbol $\not{}_i$ stands for the omission of the i-th factor, then each field ψ_i satisfies its particular first-order Dirac equation by virtue of its definition and eq. (2). The orthogonality of the projection operators (when operating on ψ) follows similarly from eq. (2), while the requirement $P_i\psi_i = \psi_i$ serves to determine the coefficients c_i; if $\delta\psi_i = -m_i\psi_i$, then

$$P_i\psi_i = c_i(m_1 - m_i)(m_2 - m_i) \dots \not{}_i \dots (m_n - m_i)\psi_i ,$$

or

$$c_i^{-1} = (m_1 - m_i)(m_2 - m_i) \dots \not{}_i \dots (m_n - m_i) .$$

The interesting algebraic properties of the c_i may be obtained most simply from the alternate expression

$$c_i = -(2\pi i)^{-1} \oint_{\varepsilon_i \to 0} \mathrm{d}z[(m_1 - z)(m_2 - z) \dots (m_n - z)]^{-1} , \tag{4}$$

where the contour of integration in the complex z plane is specified as a circle of infinitesimal radius surrounding the real pole m_i. Since the integrand of eq. (4) is analytic except at the points m_j, this contour may be arbitrarily deformed provided only that none of the other poles are included. The quantity $\sum_{i=1}^{n} c_i$ may then be expressed as the integral of the same integral and around a contour C which encloses all the singularities, and since C is arbitrary we may take it to be infinite, in which case it is clear that this integral vanishes. Proceeding in a similar fashion, we may write

$$m_i^j c_i = -(2\pi i)^{-1} \oint_{\varepsilon_i \to 0} \mathrm{d}z z^j[(m_1 - z)(m_2 - z) \dots (m_n - z)]^{-1} ,$$

and immediately conclude that

$$\begin{cases} \sum_{i=1}^{n} m_i^j c_i = 0 , & n - 1 > j \geqslant 0; \\ \sum_{i=1}^{n} m_i^{n-1} c_i = (-1)^{n-1} . \end{cases} \tag{5}$$

To verify the relation $\sum_{i=1}^{n} P_i = 1$, it is convenient to express P_i in the form

$$P_i = c_i(\delta + m_i)^{-1}[\Pi(\delta + m)]$$

and to introduce the quantity

$$b_i(\xi) = -(2\pi i)^{-1} \oint_{\varepsilon_i \to 0} \mathrm{d}z\,(\xi - z)^{-1}[(m_1 - z)(m_2 - z) \ldots (m_n - z)]^{-1} = c_i(\xi - m_i)^{-1}\,.$$

Evidently,

$$\sum_{i=1}^{n} b_i(\xi) = -[(m_1 - \xi)(m_2 - \xi) \ldots (m_n - \xi)]^{-1} = -[\Pi\,(m - \xi)]^{-1}\,,$$

and therefore

$$\sum_{i=1}^{n} P_i = -[\Pi\,\delta + m)] \sum_{i=1}^{n} b_i(-\delta) = 1\,.$$

Returning to the calculation of $S(x - x')$, we define the quantity

$$S_1(x - x') = i\,\{(\delta + m_2) \ldots (\delta + m_n)\,\psi(x), \overline{\psi}(x')\};$$

Eqs. (2) and (3) then require

$$(\delta + m_1)S_1 = 0\,, \qquad S_1|_{x_0 = x_0'} = i\gamma_4\,\delta\,(\boldsymbol{x} - \boldsymbol{x}')\,,$$

implying that S_1 is just the usual fermion anticommutator carrying the mass label m_1,

$$S_1(x - x') = S(x - x';\, m_1)\,. \tag{6}$$

Because of eq. (6) we may write

$$S_1(x - x')(\underset{\leftarrow}{\delta}' - m_1) = 0\,,$$

or

$$(\delta + m_2)\,[-i\{(\delta + m_3) \ldots (\delta + m_n)\,\psi(x), \overline{\psi}(x')(\underset{\leftarrow}{\delta}' - m_1)\}] = 0\,. \tag{7}$$

If the quantity within the square bracket of eq. (7) is denoted by $S_2(x - x')$, eqs. (3) and (7) require that

$$(\delta + m_2)\,S_2 = 0\,, \qquad S_2|_{x_0' = x_0} = i\gamma_4\,\delta(\boldsymbol{x} - \boldsymbol{x}')\,,$$

i.e., $S_2 = S(x - x';\, m_2)$. This construction may obviously be extended to demonstrate that each of the n terms

$$S_i = (-1)^{i-1}(\delta + m_n) \ldots (\delta + m_{i+1})\,S(x - x')(\underset{\leftarrow}{\delta}' - m_1) \ldots (\underset{\leftarrow}{\delta}' - m_{i-1}),$$

is identical to the corresponding anticommutator, $S_i = S(x-x';\, m_i)$.

It is then apparent that the function

$$S(x-x') = \sum_{i=1}^{n} c_i\, S(x-x';\, m_i) \tag{8}$$

satisfies each of the conditions of eq. (3) as well as the differential equation for S (the initial conditions of footnote [4] are satisfied when use is made of eq. (5)). It follows that eq. (8) represents the unique solution desired, and we may directly infer anticommutation properties of the fields ψ_i:

$$i\,\{\psi_i(x), \overline{\psi}_j(x')\} = P_i S(x-x')\gamma_4 P_j^{+\prime}\gamma_4 = c_i\delta_{ij}\, S(x-x';\, m_i)\,. \tag{9}$$

Equation (9) states that each of the ψ_i may be treated as dynamically independent; but because some of the coefficients c_i must be negative in order to satisfy eq. (5), the anticommutators of the corresponding ψ_i are negative. Following the method of Gupta and Bleuler [5], an indefinite metric will be introduced for such fields in the next section.

It is possible to cast the original Lagrangian, eq. (1), into a form such that the individual first-order Dirac equations and the anticommutation relations for the ψ_i results from independent variations $\delta\psi_i$; using the properties of the projection operators,

$$\begin{aligned} L &= -\sum_{i=1}^{n} \overline{\psi}_i[\Pi(\delta + m)]\psi = \\ &= -\{\overline{\psi}_1(\delta + m_1)[(\delta+m_2)\ldots(\delta+m_n)]\psi + \overline{\psi}_2(\delta+m_2)[(\delta+m_1)\ldots(\delta+m_n)]\psi + \ldots\} = \\ &= -\sum_{i=1}^{n} c_i^{-1}\overline{\psi}_i(\delta+m_i)\psi_i = \sum_{i=1}^{n} c_i^{-1} L_i\,, \end{aligned} \tag{10}$$

where L_i is the first-order Lagrangian conventionally used to describe a fermion of mass m_i; $L' = \sum_{i=1}^{n} L_i$ is the usual Lagrangian describing a collection of n non-interacting fermions. This procedure may be reversed: starting from eq. (10) with independent variations $\delta\psi_i$, one may construct the original ψ provided that the c_i chosen fulfil the conditions of eq. (5).

The generators of infinitesimal translations and Lorentz and gauge transformations of the field ψ, in terms of the fields ψ_i, can be read off immedia-

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[5] S. N. Gupta: *Proc. Phys. Soc.*, **63**, 681 (1950); **64**, 850 (1951); K. Bleuler: *Helv. Phys. Acta*, **23**, 567 (1950).

tely from eq. (10); in particular,

$$H = \sum_{i=1}^{n} c_i^{-1} H_i \,, \qquad J_\mu = \sum_{i=1}^{n} c_i^{-1} J_{\mu i} \,, \tag{11}$$

where H_i and $J_{\mu i}$ represent the conventional expressions for the energy and current of the field ψ_i (where, however, the ψ_i satisfy the anticommutation relations of eq. (9)).

3. – Role of the indefinite metric.

It is convenient to redefine the fields ψ_i appearing in the Lagrangian of Eq. (10) so that the coefficients c_i each have the value ± 1; such « wave-function renormalization » is trivially performed by replacing each ψ_i by $|c_i|^{\frac{1}{2}}\psi_i$. In order to satisfy eq. (9), the Fourier expansion of each ψ_i may be taken as identical to that of the corresponding conventional expression ([6]),

$$\psi_i(x) = (2\pi)^{-\frac{3}{2}} \int \mathrm{d}^3 p \left[\frac{m_i}{E_i}\right]^{\frac{1}{2}} \sum_{s=1}^{2} \{b_i(\boldsymbol{p}, s)\, u_s(\boldsymbol{p}; m_i) \exp[iP_i \cdot x] + \\ + d_i^+(\boldsymbol{p}, s) v_s(\boldsymbol{p}; m_i) \exp[-iP_i \cdot x]\} \,,$$

if the creation and destruction operators fulfil the requirement

$$\{b_i(\boldsymbol{p}, s), b_j^+(\boldsymbol{p}', s')\} = \{d_i(\boldsymbol{p}, s), d_j^+(\boldsymbol{p}', s')\} = c_i \delta_{ij}\, \delta(\boldsymbol{p} - \boldsymbol{p}')\, \delta_{ss'} \,, \tag{12}$$

with all other anticommutators vanishing.

If we retain the usual conventions where $b^+(\boldsymbol{p}, s)(d^+(\boldsymbol{p}, s))$ is the creation operator of particle (anti particle) with momentum p and spin s (nothing is gained by interchanging creation and destruction operators), we expect to define the state of the i-th particle as $|\boldsymbol{p}, s\rangle_i = b_i^+(\boldsymbol{p}, s)|0\rangle$, aside from a possibly complex normalization factor; $|0\rangle$ is understood to be the vacuum state satisfying $b_i|0\rangle = d_i|0\rangle = 0$, $\langle 0|0\rangle = 1$. The norm of such a one particle state for that field ψ_i where $c_i < 0$, would, because of eq. (12), be negative. To avoid

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([6]) See for example S. S. Schweber, H. A. Bethe and F. de Hoffmann: *Mesons and Fields*, vol. **1** (New York, 1955). The symbol $p_i \cdot x$ here stands for $\boldsymbol{p} \cdot \boldsymbol{x} - E_i x_0$, with $E_i = [\boldsymbol{p}^2 + m_i^2]^{\frac{1}{2}}$.

this, we introduce a metric operator η with the properties

$$(13)\qquad \begin{cases} \eta = \prod_{i=1}^{n} \eta_i\,, & [\eta_i, \eta_j] = 0; \\ \eta_i = \eta_i^{+} = \eta_i^{-1}, & \eta_i^2 = 1\,; \\ \eta_i \psi_j \eta_i = \psi_j\,, & i \neq j\,; \\ \eta_i \psi_i \eta_i = c_i \psi_i\,, & c_i = \pm 1; \end{cases}$$

and define the operation of taking physical matrix elements as $\langle a|\eta Q|b\rangle$. All statements concerning hermiticity properties of operators are understood to be inferred from the conventional matrix elements, $\langle a|Q|b\rangle$. In particular, with $\eta|0\rangle = +|0\rangle$, the eigenvalues of the i-th metric operator η_i, which is diagonal in the representation labeled by the eigenvalues of the i-th number operator, $N_i \doteq n_i$, are given by $\eta_i \doteq (c_i)^{n_i}$.

It may be of interest to note that the η_i have an existence and meaning quite independent of their use as metric operators; they are simply the unitary operators which generate reflections in the « internal space » of the fields ψ_i, corresponding to a trivial symmetry property of the Lagrangian of eq. (10). (Rotations in this internal space do not leave the Lagrangian unaltered because of the different factors c_i, m_i.) This does not imply that the use of an indefinite metric is a consequence of symmetry transformations, or more generally of the Action Principle; *e.g.*, the assignment of reflection operators to fields with $c_i = -1$ is arbitrary. The indefinite metric is, however, consistent with the Action Principle in the sense that statements obtained concerning generators of infinitesimal transformations are unchanged. Equal-time commutation relations are, for example, conventionally inferred from the relation [7]

$$(14)\qquad \delta O = -i[O, G]\,,$$

where δO represents an infinitesimal operator whose matrix elements, $\langle a|\delta O|b\rangle$ yield a result identical to that obtained by variation of the state vectors defining the matrix element $\langle a|O|b\rangle$. With the indefinite metric, eq. (14) is replaced by

$$(15)\qquad \delta O = -i(OG - \eta G \eta O)\,,$$

where δO denotes the operator whose matrix elements, $\langle a|\eta\,\delta O|b\rangle$, are equi-

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[7] J. SCHWINGER: *Phys. Rev.*, **82**, 914 (1951).

valent to the state vector variation of $\langle a|\eta O|b\rangle$ (8). Since the generators G_i are constructed from bilinear combinations of the fields, $G_i = ic_i^{-1}\int d^3x\, \overline{\psi}_i\gamma_4\delta\psi_i$, and the statements $\{\eta_i, \psi_i\} = 0$ or $[\eta_i, \psi_i] - 0$ imply the respective relations $\{\eta_i, \delta\psi_i\} = 0$ or $[\eta_i, \delta\psi_i] = 0$, the metric operator commutes with G; hence eqs. (14) and (15) are identical.

It is now evident that the norms of all states defined in terms of the indefinite metric are positive; *e.g.*, the norm of that state corresponding to an i-th fermion (momentum and spin are suppressed), $b_i^+|0\rangle$, is given by $\langle 0|b_i\eta b_i^+|0\rangle = c_i^2\langle 0|0\rangle - 1$, with one factor of c_i coming from the (anti-) commutator of b_i and η, and the other from the anticommutation rules of eq. (12). The same result is true for states containing arbitrary mixtures of different particles. Expectation values of the Hamiltonian (omitting zero-point terms) are positive definite; essentially the metric operator provides a factor c_i to remove the c_i^{-1} appearing in eq. (11). Similar statements hold for the expectation values of the charge operator, and for all such physical operators.

The interesting situation, however, concerns the description of interactions within the framework of the multifermion theory. For definiteness, we consider the electromagnetic coupling, but the procedure is applicable to any conventional boson-fermion interaction (9). If the Lagrangian of eq. (1) is replaced by the gauge invariant form

$$L = -\overline{\psi}[\Pi(\delta - ie\,A + m)]\psi\,, \tag{16}$$

the fermion field equation then reads

$$[\Pi(\delta - ie\,A + m)]\psi = 0\,,$$

while the equal time anticommutation rules are similar to those of eq. (3) except that each factor of δ is replaced by $\delta - ie\,A \equiv D$. Projection operators,

$$P_i^e = c_i(D + m_1)(D + m_2)\ldots\not{\ }_i\ldots(D + m_n)\,,$$

may be constructed with the same coefficients c_i, and the decomposition into fields ψ_i defined by $\psi_i = P_i^e\psi$, such that $(D + m_i)\psi_i = 0$. The Lagrangian of

(8) It is assumed that η is not a dynamical variable and is not to be varied.

(9) One merely replaces each factor of $\delta - ieA$ in eq. (16) by $\delta - g\Phi$, $\delta - ig\gamma_5\Phi$, etc.

eq. (16) may then be written in the form

$$L = -\sum_{i=1}^{n} c_i^{-1}\overline{\psi}_i(D + m_i)\psi_i \,. \tag{17}$$

Because the quantities $\{\psi_i(x), \overline{\psi}_j(x')\}$ are now no longer c-numbers, the previous free-field discussion is not directly applicable; but it is sufficient to consider independent variations $\delta\psi_i$ which lead to anti-commutation relations identical to the equal-time statements of eq. (9). Hence the factors c_i again appear in the commutation rules, and the metric operator will be needed to provide positive probabilities.

With each c_i understood to be ± 1, we may pass to the interaction representation such that the interaction Hamiltonian is

$$H' = -\int \mathrm{d}^3 x J_\mu A_\mu \,,$$

and the fermion current is given by the corresponding expression of eq. (11); all operators satisfy their respective free-field equations. The point of interest here is the sign of any given S-matrix element relative to that obtained in a conventional theory. Consider first a matrix element corresponding to, among other possible processes, a scattering of a fixed number of particles n_i of the i-th fermion field. The initial and final states are then proportional (symbolically) to $(b_i^+)^{n_i}|0\rangle$. If the i-th current operator enters m_i times, we need compute (again symbolically) the matrix element $\langle 0|(b_i)^{n_i}{}_l[c_i\overline{\psi}_i\psi_i]^{m_i}(b_i^+)^{n_i}|0\rangle$. If l_i factors ψ_i, $\overline{\psi}_i$ are paired, then $n_i = m_i - l_i$; and since each propagator carries the factor c_i, the total matrix element will be proportional to $c_i^{m_i+n_i+l_i+2n_i} = +1$. Thus, for pure scattering processes, each S-matrix element is the same as in the conventional theory. If, however, N_i pairs are produced (destroyed), the final (initial) state will contain $n_i + 2N_i$ fermions, and the same analysis shows that the transition amplitude is now given by $(c_i)^{N_i}$ times the conventional S-matrix element. Since this phase factor is independent of the order of the perturbation calculation and matrix elements corresponding to the same N_i are to be added coherently, the transition probabilities for such processes are exactly the same as in the conventional theory.

It should perhaps be remarked that the equivalence of the multifermion theory with the conventional formulation does not, apparently, extend to the equal mass situation. For example, starting from the Lagrangian of eq. (1) with $n = 2$ and both masses equal, one may construct a plausible one-fermion theory by adopting the indefinite metric together with certain subsidiary conditions defining physically acceptable state vectors (the first order Dirac equa-

tion is regarded as a subsidiary condition rather than a field equation). Interactions, and in particular production processes, spoil the consistency of such a formulation.

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It is a pleasure to acknowledge stimulating discussions with Professor R. FINKELSTEIN and Dr. D. PURSEY.

RIASSUNTO (*)

Il problema delle norme negative, che sorge in una teoria multifermionica derivata da un unico lagrangiano contenente derivate di ordine superiore al primo, può essere eliminato con l'adozione di una opportuna metrica indefinita. Si dimostra che gli elementi di interazione della matrice S, calcolati con l'operatore metrico, sono equivalenti a quelli della teoria convenzionale.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

A Model for Double Photoproduction of Charged Pions - I.

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(ricevuto il 2 Agosto 1960)

Summary. — Double photoproduction of charged pions is studied taking into account only a diagram with only one exchanged pion. Under the assumption of a sharp pion-pion resonance in the state $J=1$, $T=1$, making use of an approximate solution for the invariant scattering amplitude F, the energy spectrum of the final protons is calculated at various angles. We hope that such a theory will give at least qualitatively trustworthy results at small momentum transfers of the proton; in a subsequent paper also $\frac{3}{2}\,\frac{3}{2}$ final state state interaction will be taken into account. Calculations are made in view of a possible experiment using Frascati Synchrotron 900 MeV γ.

1. – Introduction.

Double photoproduction of charged pions on hydrogen

$$\gamma + p \xrightarrow{F} p + \pi^+ + \pi^- \tag{1}$$

may be studied taking into account diagrams of the type drawn in Fig. 1. If we take account only of those diagrams having one intermediate pion, conservation laws limit the choice to the diagram of Fig. 2. The assumption that the reaction (1), at small momentum transfers of the proton, is described essentially by the diagram of Fig. 2 is in need of some justification.

The principal reason for which we take into account only this diagram is to evaluate the influence of a possible pion-pion interaction in double photoproduction. In fact if we consider the diagram of Fig. 3, which describes photoproduction of pions on pion and is an essential part of the diagram of Fig. 2,

(*) This work has been done whilst the authors were at the Università di Roma with a scholarship of the Istituto Nazionale di Fisica Nucleare.

we see that it will give an appreciable contribution only in the case of a strong pion-pion interaction (1).

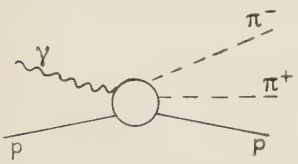

Fig. 1.

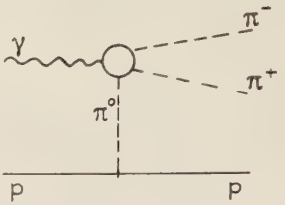

Fig. 2.

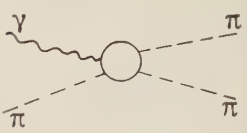

Fig. 3.

As to this interaction, there are already at least qualitative indications that it exists (2), and therefore an eventual agreement of our theory with experiment may be considered another proof of its existence. Moreover, the diagram of Fig. 2 may also be interpreted in terms of a model, as it has recently been suggested in a similar case (*), and it may be considered to describe the following physical situation: « The incident γ, instead of interacting directly with the proton, interacts with a pion of the mesonic cloud which surrounds the proton ». In any case it may be said that, also when diagrams with more than one exchanged pion are considered, the diagram of Fig. 2 must remain the most important at small momentum transfers of the proton, and this is due to the smallness of the denominator $(\Delta^2+\mu^2)^2$, where $(\Delta^2+\mu^2)^{-1}$ represents the propagator of the intermediate pion. Therefore it seems sufficiently justified to think that the energy spectrum of small energy protons may be calculated, at least within an order of magnitude starting from the diagram of Fig. 2.

Another support to this hypothesis may be considered the fact that, at $\Delta^2=-\mu^2$, *i.e.* in the non-physical region where the intermediate pion is on its mass shell, the theory becomes exact, as CHEW and LOW have pointed out (3).

(1) M. GOURDIN and A. MARTIN: *Nuovo Cimento*, **16**, 78 (1960).

(2) F. J. DYSON: *Phys. Rev.*, **99**, 1037 (1955); G. TAKEDA: *Phys. Rev.*, **100**, 440 (1955); M. CINI and A. EBERLE: *Proc. of Intern. Conf. on Mesons and Newly Discovered Particles* (Padua-Venice, 1957); W. G. HOLLADAY: *Phys. Rev.*, **101**, 1198, 1202 (1956); W. R. FRAZER and J. R. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959); *Phys. Rev.* **117**, 1609 (1960); J. BOWCOCK, W. N. COTTINGHAM and D. LURIÉ: *Nuovo Cimento*, **16**, 918 (1960); F. CERULUS: *Nuovo Cimento*, **14**, 827 (1959); L. BROWN and F. CALOGERO: *Phys. Rev. Lett.*, **4**, 315 (1960); I. DERADO: *Nuovo Cimento*, **15**, 853 (1960); L. BROWN and F. CALOGERO: *Phys. Rev. Lett.* (to be published); N. CABIBBO and R. GATTO: *Phys. Rev. Lett.*, **4**, 313 (1960). Further references on π-π interaction may be found in C. J. GOEBEL: *Phys. Rev. Lett.*, **1**, 337 (1958) and in P. CARRUTHERS and H. A. BETHE: *Phys. Rev. Lett.*, **4**, 536 (1960).

(*) See, for the case of pion production in π-nucleon scattering the paper by BONSIGNORI and SELLERI (2).

(3) G. F. CHEW and F. E. LOW: *Phys. Rev.*, **113**, 1640 (1959).

But it must be pointed out that the results we have obtained may be partially modified by final state interactions, principally by π^+-proton interaction, being these two particles in a pure $\frac{3}{2}\,\frac{3}{2}$ state. This interaction may be taken into account in two ways: analysing the matrix element in partial waves and extracting the part containing the final π^+ and proton in a P wave and then evaluating the correction to cross-section; or studying the diagram in which the π^+-proton resonant state is represented by a particle of suitable masse. The first of these corrections will be calculated in a subsequent paper.

2. – Calculation of partial cross sections.

Considering the diagram of Fig. 2, we define the four-momenta:

K of the incident photon,
P_1 of the initial proton,
q_1 of the intermediate pion,
q_2 of the final π^-,
q_3 of the final π^+,
P_2 of the final proton.

The matrix element R for the reaction (1) then may be written

$$\langle q_2 q_3 P_2 | R | K P_1 \rangle = \delta^4(q_2 + q_3 + P_2 - K - P_1) \langle q_2 q_3 P_2 | M | K P_1 \rangle , \tag{2}$$

where

$$\langle q_2 q_3 P_2 | M | K P_1 \rangle = \frac{1}{2i(2\pi)^{7/2}} \frac{m}{\sqrt{P_{10} P_{20}}} \frac{\varepsilon_{\lambda\mu\nu\varrho}\, q_1^\mu q_2^\nu q_3^\varrho e^\lambda}{\sqrt{8 K_0 q_{20} q_{30}}} \frac{F}{(q_1^2 + \mu^2)} \cdot g \cdot \bar{u}(\boldsymbol{P}_2) \gamma_5 u(\boldsymbol{P}_1) ,$$

and e^λ is the polarization vector of the photon, F a scalar amplitude of which one can give a spectral representation, and g is the usual coupling constant of the meson theory. The matrix element (2) has been written using the matrix element of the photoproduction of pions on pion (*) and assuming:

a) use of the propagator $(q_2^1 + \mu^2)^{-1}$ for the intermediate pion, instead of the correct propagator;

b) use of the vertex function γ_5 instead of the renormalized one.

These hypothesis are equivalent to suppose that only diagrams with only one exchanged pion contribute (4): therefore all the assumptions we have made are of the same order of approximation.

The scalar amplitude F is a function of three invariants S_1, S_2, S_3 defined

(*) See Gourdin and Martin (1).
(4) F. Bonsignori and F. Selleri: *Nuovo Cimento*, **15**, 465 (1960).

for the reaction

$$\gamma + \pi \to \pi + \pi \tag{3}$$

in the following way

$$\begin{cases} S_1 = -(q_2 + q_3)^2 , \\ S_2 = -(q_1 + q_3)^2 , \\ S_3 = -(q_1 + q_2)^2 . \end{cases} \tag{4}$$

A Mandelstam representation containing a unique weight function is assumed for $F(S_1, S_2, S_3)$. GOURDIN and MARTIN, following the Cini-Fubini method ([5]) which reduces the spectral representation of F to

$$F(S_1, S_2, S_3) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(S')\,\mathrm{d}S'}{S' - S_1} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(S')\,\mathrm{d}S'}{S' - S_2} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{\varrho(S')\,\mathrm{d}S'}{S' - S_3} , \tag{5}$$

have given an approximate solution of F assuming a sharp pion-pion resonance. This solution contains a multiplicative constant not yet determined.

Such a constant, $\varphi(-\frac{3}{4})$ in the paper by GOURDIN and MARTIN, will be used by us in the adimensional form $\gamma = \mu^3\varphi(-\frac{3}{4})$, where μ is the pion mass. Further on in this paper, we will write the approximate solution of $F(S_1, S_2, S_3)$ as $(\gamma/\mu)F(w^2)$, where $w^2 = S_1$; in the calculations of the energy spectrum of the final protons we will use both $F(w^2)$ calculated in the case of a pion-pion resonance of zero width and the one calculated using the actual width (*).

The cross-section is given by

$$\mathrm{d}\sigma = (2\pi)^2 S_f \bar{S}_i \delta^4(q_2 + q_3 + P_2 - P_1 - K)\,|\langle q_2 q_3 P_2 |\, M \,| K P_1\rangle|^2 \cdot \frac{K_0 P_{10}}{(KP_1)} . \tag{6}$$

Summing and averaging over spins and summing over polarizations, we have

$$\bar{S}_i \sum_S |\,\bar{u}(\boldsymbol{P}_2)\gamma_5 u(\boldsymbol{P}_1)\,|^2 = \frac{-(P_1P_2) - m^2}{2m^2} ,$$

$$\sum_i \Big|\frac{1}{i}\varepsilon_{\lambda\mu\nu\varrho} q_1^\mu q_2^\nu q_3^\varrho e_{(i)}^\lambda\Big|^2 = -\varepsilon_{\lambda\mu\nu\varrho}\varepsilon_{\lambda'\mu'\nu'\varrho'} q_1^\mu q_1^{\mu'} q_2^\nu q_2^{\nu'} q_3^\varrho q_3^{\varrho'} \cdot \sum_i e_{(i)}^\lambda e_{(i)}^{\lambda'} =$$

$$= -\varepsilon_{\lambda\mu\nu\varrho}\varepsilon_{\lambda'\mu'\nu'\varrho'} K^\mu K^{\mu'} q_2^\nu q_2^{\nu'} q_3^\varrho q_3^{\varrho'} \delta_{\lambda\lambda'} = -\varepsilon_{\lambda\mu\nu\varrho}\varepsilon_{\lambda'\mu'\nu'\varrho'} K^\mu K^{\mu'} q_2^\nu q_2^{\nu'} q_3^\varrho q_3^{\varrho'} =$$

$$= -(\delta_{\mu\mu'}\delta_{\nu\nu'}\delta_{\varrho\varrho'} - \delta_{\mu\nu'}\delta_{\nu\mu'}\delta \quad - \delta_{\mu\varrho'}\delta_{\nu\nu'}\delta_{\varrho\mu'} + \delta_{\mu\nu'}\delta_{\nu\varrho'}\delta_{\varrho\mu'} -$$

$$- \delta_{\mu\mu'}\delta_{\nu} \quad \delta_{\varrho\nu'} + \delta_{\mu\varrho'}\delta_{\nu\mu'}\delta_{\varrho\nu'}) K^\mu K^{\mu'} q_2^\nu q_2^{\nu'} q_3^\varrho q_3^{\varrho'} =$$

$$= q_3^2 (Kq_2)^2 + q_2^2 (Kq_3)^2 - 2(Kq_2)(kq_3)(q_2 q_3) .$$

([5]) M. CINI and S. FUBINI: *Ann. Phys.* **3**, 352 (1960).

(*) See GOURDIN and MARTIN ([1]), Appendix.

Therefore

$$\mathrm{d}\sigma = \frac{1}{8(2\pi)^5} g^2 \frac{\gamma^2}{\mu^6} \int \mathrm{d}\tau_f \, \delta^4(q_2 + q_3 + P_2 - P_1 - K) Q \,, \tag{7}$$

where

$$Q = \frac{1}{8(KP_1) q_{30} q_{20} P_{20}} \frac{[-(P_1 P_2) - m^2]}{(q_1^2 + \mu^2)^2} \cdot$$
$$\cdot [q_3^2 (Kq_2)^2 + q_2^2 (Kq_3)^2 - 2(Kq_2)(Kq_3)(q_2 q_3)] \cdot F^2(w^2) \,,$$

and

$$\mathrm{d}\tau_f = \mathrm{d}^3 P_2 \, \mathrm{d}^3 q_2 \, \mathrm{d}^3 q_3 \,.$$

Rewriting all that in an invariant form:

$$\int \mathrm{d}\tau_f \, \delta^4(q_2 + q_3 + P_2 - P_1 - K) Q = \tag{8}$$
$$= \int \mathrm{d}^4 P_2 \, \delta(P_2^2 + m^2) \, \mathrm{d}^4 q_1 \, \delta^4(P_1 - P_2 - q_1) \frac{[-(P_1 P_2) - m^2]}{(KP_1)(q_1^2 + \mu^2)^2} \cdot$$
$$\cdot \int \mathrm{d}^4 q_2 \, \mathrm{d}^4 q_3 \, \delta(q_2^2 + \mu^2) \, \delta(q_3^2 + \mu^2) \, \delta^4(q_3 + q_2 - q_1 - K) g[(Kq_2), (Kq_3), (q_2 q_3)] \cdot F^2(w^2) \,.$$

Making use of conservation laws, we obtain

$$g[(Kq_2), (Kq_3), (q_2 q_3)] \to g'[(Kq_1), K(q_2 - q_3), (q_2 q_3)] =$$
$$= -\{\tfrac{1}{4}\mu^2 [(Kq_1) + K(q_2 - q_3)]^2 + \tfrac{1}{4}\mu^2 [(Kq_1) - K(q_2 - q_3)]^2 +$$
$$+ \tfrac{1}{2}(q_2 q_3) \cdot [(Kq_1) + K(q_2 - q_3)][(Kq_1) - K(q_2 - q_3)]\} \,.$$

In this way we can calculate very easily the second integral of (8), evaluating it in the c.m. system of the two final pions. We obtain

$$\int \mathrm{d}^4 q_2 \, \mathrm{d}^4 q_3 \, \delta(q_2^2 + \mu^2) \, \delta(q_3^2 + \mu^2) \, \delta^4(q_2 + q_3 - q_1 - K) g'[(Kq_1), K(q_2 - q_3), (q_2 q_3)] F^2(w^2) = \tag{9}$$
$$= -\frac{\pi/2}{K_0 w} \left\{ \frac{1}{4} \mu^2 [(Kq_1) + X]^2 + \frac{1}{4} \mu^2 [(Kq_1) - X]^2 + \right.$$
$$\left. + \frac{1}{2} \left(\mu^2 - \frac{w^2}{2} \right) [(Kq_1) + X][(Kq_1) - X] \right\} F^2(w^2) \, \mathrm{d}X \,,$$

where the invariant X is defined by

$$X = K(q_2 - q_3) \tag{10}$$

and, in the c.m. system of the two final pions, is

$$X = 2K_0 q \cos\varphi ,$$

being φ the angle between $\boldsymbol{K}$ and $\boldsymbol{q}_2$ and q the common modulus of $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$.

Now we integrate the first part of (8) and pass to the variables Δ^2 and w^2, where:

$$\Delta^2 = (P_1 - P_2)^2 = 2mT_{2L} ,$$

$$w^2 = -(q_2 + q_3)^2 = T_{2L}^2 - 2K_{0L}(T_{2L} - P_{2L}\cos\theta_L) - P_{2L}^2 ,$$

being: T_{2L}, P_{2L} kinetic energy and momentum of the final proton in the lab. system;

θ_L the angle between the direction of the final proton and forward direction;

K_{0L} the energy of the incident γ in the lab. system.

One has, evaluating the integral in the lab. system,

$$(11) \qquad \int \mathrm{d}^4P_2\,\delta(P_2^2 + m^2)\,\mathrm{d}^4q_1\,\delta^4(P_1 - P_2 - q_1)\,\frac{-(P_1P_2) - m^2}{(KP_1)(q_1^2 + \mu^2)^2} = \frac{\pi}{4m^2K_{0L}^2}\,\frac{\frac{1}{2}\Delta^2}{(\Delta^2 + \mu^2)^2}\,\mathrm{d}\Delta^2\,\mathrm{d}w^2 .$$

Expressing also (Kq_1) as a function of w^2 and Δ^2, from (9) and (11) one has finally

$$(12) \qquad \frac{\partial\sigma}{\partial\Delta^2\,\partial w^2\,\partial X} = \frac{4.84\cdot10^{-4}}{(2\pi)^3}\,g^2\,\frac{\gamma^2}{\mu^6}\,\frac{\Delta^2}{(\Delta^2 + \mu^2)^2}\left[\left(\frac{w^2}{4} - \mu^2\right)(w^2 + \Delta^2)^2 - w^2X^2\right]\frac{F^2(w^2)}{wK_0m^2K_{0L}^2} .$$

Eq. (12) gives the cross-section for the reaction (1) as a function of the energy and angle of the proton and of the angle between π^- direction and forward direction; integrating (12) with respect to X one obtains the partial cross-section as a function only of the proton variables. The limits of X are $\pm 2K_0\sqrt{(w^2/4) - \mu^2}$ and the integration of (12) gives immediately

$$(13) \qquad \frac{\partial\sigma}{\partial\Delta^2\,\partial w^2} = \frac{19.36\cdot10^{-4}}{(2\pi)^3}\,g^2\,\frac{\gamma^2}{\mu^6}\,\frac{F^2(w^2)}{m^2K_{0L}^2w}\,\frac{\Delta^2}{(\Delta^2 + \mu^2)^2}\left(\frac{w^2}{4} - \mu^2\right)^{\frac{3}{2}}\left[(w^2 + \Delta^2)^2 - \frac{4}{3}K_0^2w^2\right] .$$

From (13), considering γ-rays of given energy, it is possible to calculate the energy of the protons at various angles; such a calculation has been made considering γ of 900 MeV, taking into consideration the possibilities of the Frascati Synchrotron, and the results will be discussed in the following section.

3. – Discussion of the results.

For a numerical calculation (13), now it is necessary to make use of the approximate solution of $F(w^2)$. As we have already said, we will use both the zero width solution and the one constructed by suitable parameters for an actual width; both these solutions are taken from the Gourdin-Martin paper.

In our notations we will have (now $\mu = 1$ units are used)

$$F^2(w^2) = \frac{256}{(20 - w^2)^2}\left(\frac{39 - w^2}{37 + w^2}\right)^2, \tag{14}$$

in the case of zero width, and

$$F^2(w^2) = \frac{109.42 + 13.15w^2}{(7.97 + 0.25w^2)(0.062w^4 - 2.37w^2 + 26.56)}\left(\frac{60.4 - w^2}{57.4 + 2w^2}\right)^2, \tag{15}$$

in the case of the actual width.

(14) corresponds to $B(\nu)$ by GOURDIN and MARTIN with $\nu_R = 4$ and (15) to $B(\nu)$ with the parameters given by the same authors in the Appendix of their paper; remember that $\nu = (w^2/4) - 1$.

Inserting (14) in (13), let us calculate $\partial\sigma/\partial\Delta^2$ at the following angles of

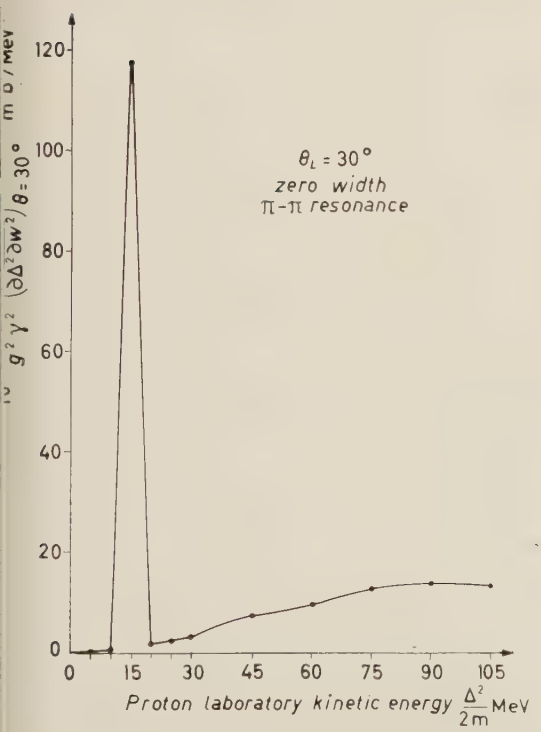

Fig. 4.

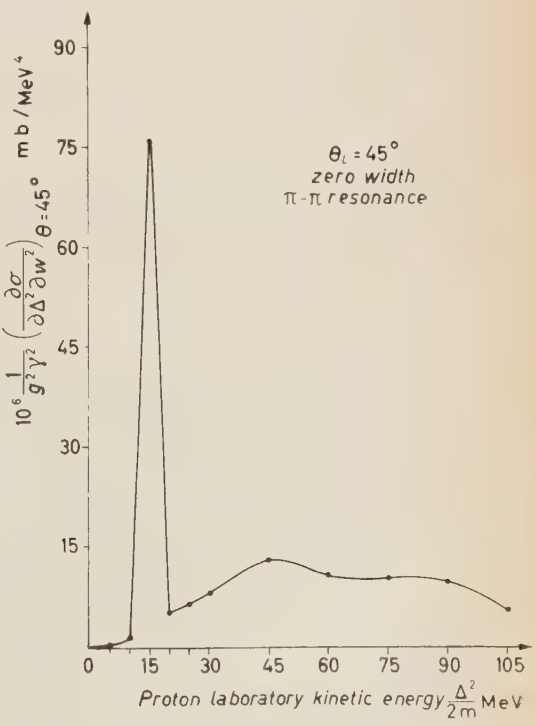

Fig. 5.

the proton: $\theta_L = 30°$, $45°$; the graphs we have obtained are drawn in Fig. 4 and 5. Using now the actual width solution (15), we obtain for $\theta_L = 15°$ $30°$, $45°$ the graphs drawn in Fig. 6, 7, 8.

A detailed examination of the terms which contribute to $\partial\sigma/\partial\Delta^2$ shows that the peak which appears at 15 MeV of proton kinetic energy is essentially due

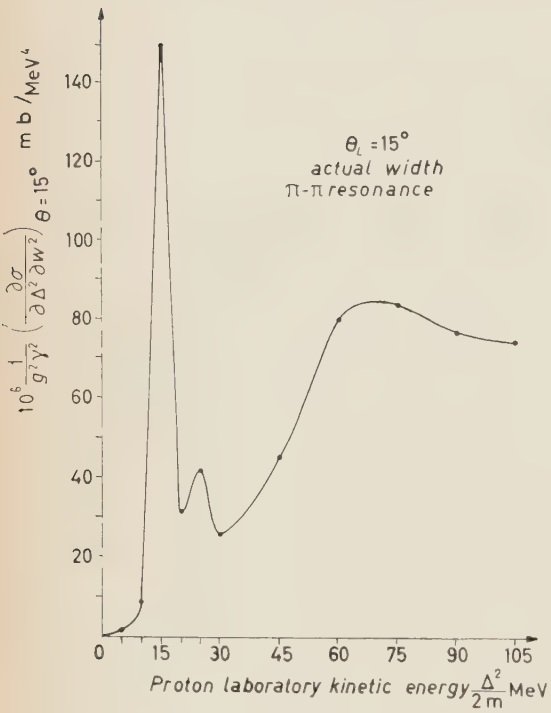

Fig. 6.

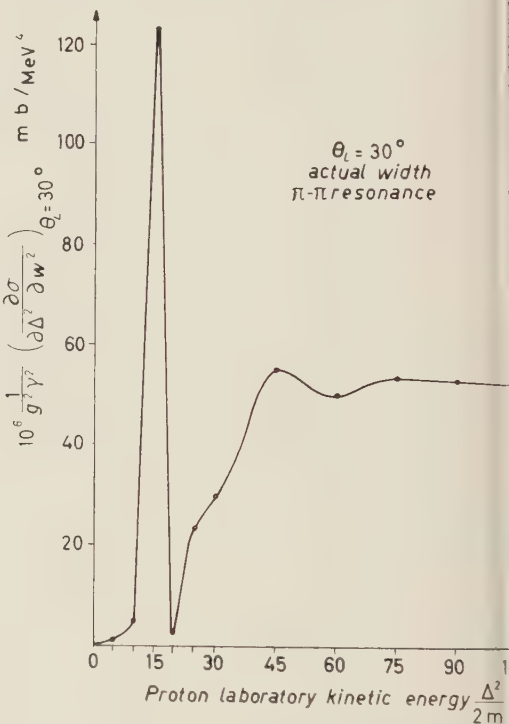

Fig. 7.

to the denominator $(\Delta^2+\mu^2)^2$ combined with the behaviour of w^2: *i.e.* this peak is due to the fact that we have considered the diagram of Fig. 2, with only an exchanged pion.

At $\theta_L = 30°$, $45°$ the invariant w^2 does not reach the value 20 which would make $F^2(w^2)$ of zero width infinite: therefore at these angles the pion-pion resonance does not appreciably contribute to the cross-section.

In order that the resonance may fall into the range of proton energies allowed by the kinematics, we need γ's of a greater energy.

Comparing the Fig. 4 and 5 with Fig. 7 and 8, one sees that in this case, for the reasons given, the introduction of an actual width changes very little the form of the curves.

At $\theta_L = 15°$, on the contrary, ω^2 can reach the value 20 and then (14)

would give an infinite solution. Fig. 6 gives $\partial\sigma/\partial\Delta^2$ at 15° obtained by the actual width solution; as one can see the contribution of the pion-pion resonance which gives the second peak at about 25 MeV is very little compared with the one at 15 MeV. From this we can derive the following conclusions.

Double photoproduction of charged pions is a good experimental test for the existence of pion-pion interaction and allows us to check the goodness of the assumptions we have made in this paper, *i.e.* that, at least at small Δ^2, the reaction (1) is essentially described by the diagram of Fig. 2. But, at the γ energy we have used, one has very few information on the position and width of π-π resonance.

Moreover, from our calculations, it can be seen that $F^2(w^2)$, in our units $\mu = 1$, has a value which oscillates about 1 and the maximum oscillation is about 40%.

Roughly it can be said that the superior vertex of the diagrams of Fig. 2 can be assumed as a point interaction with a «weighted» coupling constant.

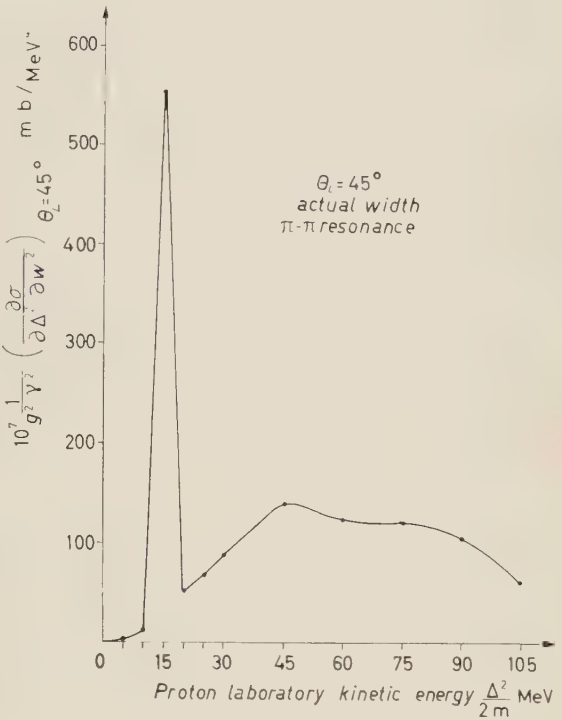

Fig. 8.

It remains still to evaluate the corrections due to the final state interactions, however there are some good reasons for thinking that, at least the most important of them (π^-p), will not appreciably change the peak at 15 MeV. We will discuss this point in the following section.

4. – A proposed experiment.

In the present state of experimental techniques, an experiment of double photoproduction as it will be necessary for an experimental verification of the graphs of Fig. 4-8 is rather difficult to perform, not having at our disposal monochromatic γ beams.

It is therefore more convenient to perform an experiment in which we detect not only the proton but also a pion, for instance the π^-, at a fixed angle and determine its energy; in this way γ's of the wanted energy can be selected *a posteriori*. The most simple kinematic situation is the one in which π^- is fixed forward: however even this situation presents remarkable difficulties.

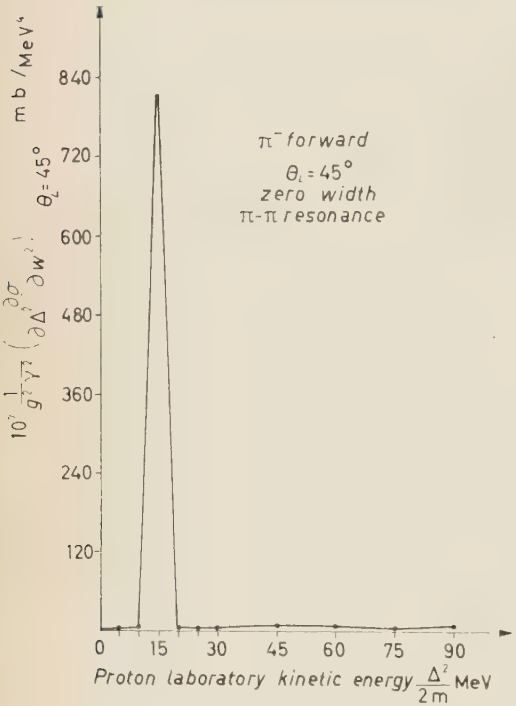

Fig. 9.

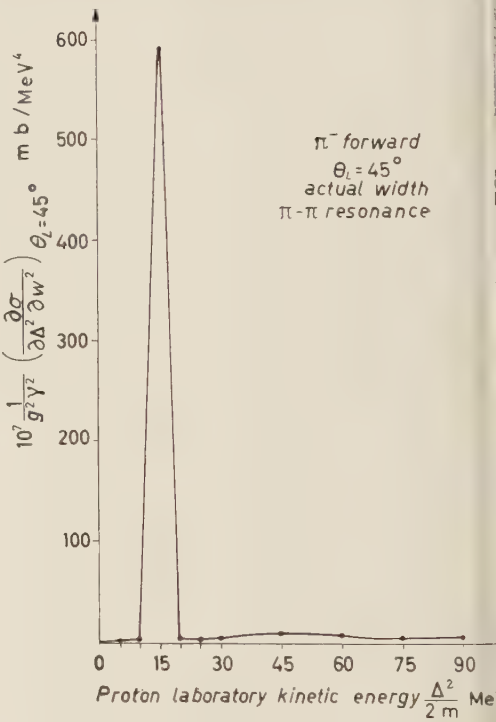

Fig. 10.

In fact the background makes it difficult to detect π^- in the direction of the beam. However we have chosen this case because of the great simplicity of the kinematics: the aim of the calculation we have made is that of seeing if the graphs of $\partial\sigma/\partial\Delta^2$ remain similar to the ones in the general case; we will see that this in fact is the case. If π^- is fixed forward, the invariant X may be evaluated very easily in the lab. system. One has

$$X = 2K_{0L}(q_{2L} - q^0_{2L}) - K_{0L}T_{2L} + K_{0L}P_{2L}\cos\theta_L\,, \tag{16}$$

where q_{2L} and q^0_{2L} are the momentum and the energy of π^- in the lab. system, and from conservation laws one has

$$q_{2L} = \frac{b(d^2 - c^2) + c\sqrt{(d^2 - c^2)^2 + 4(b^2 - c^2)}}{2(b^2 - c^2)}\,,$$

with

$$a = P_{2L} \sin \theta_L ,$$

$$b = -K_{0L} - P_{2L} \cos \theta_L ,$$

$$c = K_{0L} - T_{2L} ,$$

$$d^2 = a^2 + b^2 .$$

Inserting (16) in (12) and at $\theta_L = 45°$, we have the graphs of Fig. 9 and 10 which are obtained using the zero width solution and the actual width solution respectively. As one may see these curves are similar, at least for the principal peak, to the preceeding ones.

Evaluating the relative kinetic energy of the proton and π^+, as a function of T_{2L} and θ_L, we find that at 45° and for $T_{2L} < 30$ MeV the $\frac{3}{2}\,\frac{3}{2}$ π^+p resonance doesn't change appreciably the calculated curve.

Owing to the great similarity of all the graphs in the 15 MeV peak, one may reasonably think that, as regards the first peak, the calculated curves are not changed by the $\frac{3}{2}\,\frac{3}{2}$ final state interaction and therefore are experimentally directly verifiable.

In a subsequent paper however we will evaluate in detail the corrections for the $\frac{3}{2}\,\frac{3}{2}$ final state interaction.

* * *

We are indebted to Prof. M. Cini for having suggested this calculation and for his constant helpful advice.

We wish to thank also Drs. N. Cabibbo, H. Munczek, F. Selleri and Prof. A. Stanghellini for fruitful discussions.

RIASSUNTO

Viene studiata la fotoproduzione doppia di pioni carichi tenendo conto soltanto di un diagramma con scambio di un solo pione. Nell'ipotesi di una risonanza pione-pione nello stato $J=1$, $T=1$, facendo uso di una soluzione approssimata per l'ampiezza di scattering invariante F, viene calcolato lo spettro energetico dei protoni finali a vari angoli. Si spera che una tale teoria dia risultati almeno qualitativamente attendibili per piccoli trasferimenti d'impulso del protone; in un prossimo lavoro si terrà conto anche dell'interazione nello stato finale $\frac{3}{2}\,\frac{3}{2}$. I calcoli sono fatti in vista di un possibile esperimento col Sincrotrone di Frascati usando γ da 900 MeV.

On the Fermi-Yang Theory of the Pions (*).

K. Baumann, P. G. O. Freund and W. Thirring
Institute for Theoretical Physics, University of Vienna - Vienna

(ricevuto il 3 Agosto 1960)

Summary. — It is proposed to reduce the strong interactions to the universal weak interaction, by considering them as high energy phenomena of the strongly energy dependent weak interaction. According to Fermi and Yang the pion is assumed to be a nucleon-antinucleon bound state. A method presented in a previous paper is used to solve a Bethe-Salpeter equation. The bound state wave function is normalized, and the pion-nucleon coupling constant is calculated. The latter turns out to be of the order unity although a weak interaction between the fermions is used.

1. – Introduction.

Together with the problem whether some of the known strongly interacting particles are composite particles, the question arises if the known types of interactions (weak, electromagnetic, strong) are all fundamental or if they represent but special aspects of a more universal interaction (1). One would like to believe that the latter is the case and one is naturally tempted to consider as universal the weak universal Fermi interaction (U.F.I.). What one then has to show is that by means of this interaction the really elementary fermions may form strongly bound boson states which on their turn are capable of strong interactions with the primary fermions. This possibility may not be completely out of question since the strength of the U.F.I. is energy dependent (at high energy it becomes strong).

(*) Supported in part by the United States Government.

(1) A similar situation occured in physics when electrostatic and magnetostatic interactions proved to be limiting cases of one and the same electromagnetic interaction.

In this paper we shall not touch the question if it is possible to reduce everything to three fundamental massless Weyl fields. Instead, we start with a Dirac field which might or might not be considered as fundamental.

What we are going to do is to seek a ladder approximation solution of the Bethe-Salpeter equation of the nucleon-antinucleon interaction possessing the correct symmetry properties of the pion (using techniques developed in a previous paper [2]), and then to convince ourselves that the so constructed pion is capable of strong interactions with the nucleon. That the pion interacts strongly with the nucleons even at low energies is then physically only a consequence of the fact that the pion itself contains extreme relativistic nucleon-antinucleon pairs.

To compute the lifetime of the π-meson and convince ourselves that the $\pi \to \mu + \bar{\nu}$ decay really results to be weak would be a crucial test for the theory to be developed in this paper. This will be the subject of forthcoming investigations. Within the approximation considered here the pion lifetime is infinite.

2. – Symmetry properties.

The four-component theory we are developing in this paper is invariant under P and C separately (while the two-component theory considered in I was invariant only under CP).

The transformation laws of the fundamental four-component spinor are

$$(1) \qquad \begin{cases} P\psi(x)\,P^{-1} = \gamma_0\psi(x_r)\,, \\ C\psi(x)\,C^{-1} = C\overline{\psi}^T(x)\,, \end{cases}$$

where C is the usual charge conjugation matrix with the properties

$$(2) \qquad \begin{cases} C^{-1}\gamma_\mu C = -\gamma_\mu{}^T\,, \\ C^T = -C\,, \qquad C^+ = C^{-1}\,, \end{cases}$$

and

$$(x_r^0,\ \boldsymbol{x}_r) = (x^0,\ -\boldsymbol{x})\,.$$

Since we are interested to obtain as a bound state the pion the corresponding field $\varphi(x)$ must be pseudoscalar and possess even charge parity

$$(3) \qquad \begin{cases} P\varphi(x)\,P^{-1} = -\varphi(x_r)\,, \\ C\varphi(x)\,C^{-1} = \varphi(x)\,. \end{cases}$$

(2) K. Baumann and W. Thirring: *Nuovo Cimento*, quoted in the following as I.

For the vacuum expectation values of the time-ordered products

$$\tau_{\alpha\beta'}(x, y, z) = \langle T\,\psi_\alpha(x)\,\overline{\psi}_{\beta'}(y)\,\varphi_{l'}(z)\rangle_0 \tag{4}$$

the corresponding symmetry properties are:

$$\begin{cases} \tau(x, y, z) = -\gamma_0\,\tau(x_r, y_r, z_r)\,\gamma_0\,, \\ \tau(x, y, z) = C\,\tau^T(y, x, z)\,C^{-1}\,. \end{cases} \tag{5}$$

Defining the Bethe-Salpeter (B.S.) wave function $\varphi_q(x)$ of a pion with four-momentum q by the relation

$$\langle 0\,|\,T\psi(x)\overline{\psi}(x')\,|\pi_q\rangle = \chi_q(x, x') = \exp\left[-iq\,\frac{x+x'}{2}\right]\varphi_q(x-x')\,, \tag{6}$$

one easily finds, by means of the reduction formula, that the symmetry properties (5) of the τ-function imply

$$\begin{cases} \varphi_q(x) = -\gamma_0\,\varphi_{q_r}(x_r)\,\gamma_0\,, \\ \varphi_q(x) = C\,\varphi_q^T(-x)\,C^{-1}\,. \end{cases} \tag{7}$$

These conditions limit the form of $\varphi_q(x)$ to

$$\begin{aligned} \varphi_q(x) = {} & \gamma_5 f_1(x^2, (xq)^2) + \gamma_5(\gamma q)\,f_2(x^2, (xq)^2) + \\ & + \gamma_5(\gamma x)(qx) f_3(x^2, (xq)^2) + \gamma_5\big((\gamma x)(\gamma q) - (\gamma q)(\gamma x)\big) f_4(x^2, (xq)^2)\,. \end{aligned} \tag{8}$$

3. – B.S. equation. Pseudoscalar solution.

Using the cut-off procedure by means of an intermediate heavy vector boson of mass M in the way suggested in I the field equations write

$$(\gamma p - m)\psi = \lambda A\gamma\psi \qquad (p = i\partial) \tag{9}$$

for V coupling.

For the problem of the pion which we are interested in, ψ is the nucleon field [3] and correspondingly m the nucleon mass. The B.S. equation is

$$(\gamma p_x - m)\,\chi_q(x, y)(\gamma \overleftarrow{p}_x + m) = i\lambda^2\,\gamma_\mu\,\chi_q(x, y)\gamma^\mu\,\Delta_f(x-y, M^2)\,.$$

---

[3] As in I, we neglect the isotopic degrees of freedom.

Passing to φ_q and working out the left hand side we find the final form

$$(10) \qquad [(p+\tfrac{1}{2}q,\gamma)-m]\varphi_q(x)[(p-\tfrac{1}{2}q,\gamma)-m] = i\lambda^2 \Delta_f(x, M^2)\gamma_\mu \varphi_q(x)\gamma^\mu .$$

Since in our case

$$(11) \qquad q^2 = \mu^2 \ll m^2 \ll M^2 \qquad (\mu = \text{pion mass})$$

we may in a first approximation drop all terms containing q in $\varphi_q(x)$. In order to solve the B.S. equation we proceed like in I using for the pseudoscalar solution (8) the following spectral representation

$$(12) \qquad \varphi_q(x) = \gamma_5 \int_0^\infty \mathrm{d}c\, \tilde{\varrho}(c)\, \Delta_f(x, c) + 0(q) .$$

However, for the following it will prove useful to rewrite (12) in the form

$$(13) \qquad \varphi_q(x) = \gamma_5 \int_{-m^2}^\infty \mathrm{d}c\, \varrho(c)\, \Delta_f(x, c+m^2) , \qquad \varrho(c) = \tilde{\varrho}(c+m^2) .$$

Inserting (13) into (10) we obtain the simple integral equation

$$(14) \quad (\Box + m^2)\int_{-m^2}^\infty \mathrm{d}c\, \varrho(c)\, \Delta_f(x, c+m^2) = 4i\lambda^2 \int_{-m^2}^\infty \mathrm{d}c\, \varrho(c)\, \Delta_f(x, c+m^2)\, \Delta_f(x, M^2) .$$

Making use of the identity

$$(15) \qquad b\, \Delta_f(x, b+m^2) = -(\Box + m^2)\left[\Delta_f(x, b+m^2) - \int_{-m^2}^\infty \mathrm{d}c\, \Delta_f(x, c+m^2)\, \delta(c)\right] ,$$

and applying the techniques developed in I, the solution of (14) results of the form

$$(16) \qquad \varrho(c) = \delta(c) + \sigma(c) ,$$

with $\sigma(c)$ obeying the integral equation

$$(17') \qquad \sigma(c) = -f_0(c, 0) - \int_{-m^2}^\infty \mathrm{d}a\, f_0(c, a)\, \sigma(a) ,$$

and the condition

$$1+\int_{-m^2}^{\infty} \mathrm{d}c\,\sigma(c)=0\,, \tag{17''}$$

where

$$f_0(c,a)=\frac{\lambda^2}{4\pi^2}\,\frac{c+a+2m^2-M^2}{c(c+m^2)}\cdot \tag{17'''}$$

$$\cdot\frac{1}{\sqrt{(c+a+2m^2-M^2)^2-4(a+m^2)(c+m^2)}}\,\theta\big(\sqrt{c+m^2}-\sqrt{a+m^2}-M^2\big)\,.$$

Here like in I, we have made the « regularization » $\Delta_f(x,M^2)\to-\partial/\partial M^2\cdot\Delta_f(x,M^2)$ in order to avoid divergence difficulties at the calculation of λ. The first Fredholm approximation to (17') is

$$\sigma(c)=-f_0(c,0)=-\left(\frac{\lambda}{2\pi M^2}\right)^2\frac{c/M^2-1+2\varepsilon}{c/M^2(c/M^2+\varepsilon)}\cdot \tag{18}$$

$$\cdot\frac{1}{\sqrt{(c/M^2-1+2\varepsilon)^2-4c\varepsilon}}\,\theta\left(\frac{c}{M^2}-1-2\sqrt{\varepsilon}\right),$$

where the notation ε has been introduced for the small quantity m^2/M^2.

Inserting (18) into (17'') we find:

$$\lambda^2=(2\pi M^2)(1-\sqrt{\varepsilon}+0\,(\sqrt{\varepsilon})) \tag{19}$$

thus a value very near to the one obtained in the two-components theory in I.

Inserting (16), (18), (19) into (12) we find for the wave function $\varphi_a(x)$ with the above mentioned approximations

$$\varphi_a(x)=N\gamma_5\Big[\Delta_f(x,m^2)-(1-\sqrt{\varepsilon})\cdot \tag{20}$$

$$\cdot\int_{1+2\sqrt{\varepsilon}}^{\infty}\frac{\mathrm{d}c}{c(c+\varepsilon)}\,\frac{c-1+2\varepsilon}{c-1-2\varepsilon/(c-1)}\,\Delta_f(x,(c+\varepsilon)M^2)\Big]\,,$$

where N is a normalization factor, the exact value of which will be determined in the next section.

From (20) one also obtains

$$\varphi_a(p)=N\gamma_5\Big[\frac{1}{p^2-m^2}-(1-\sqrt{\varepsilon})\int_{1+2\sqrt{\varepsilon}}^{\infty}\frac{\mathrm{d}c}{c(c+\varepsilon)}\,\frac{c-1+2\varepsilon}{c-1-2\varepsilon/(c-1)}\,\frac{1}{p^2-(c+\varepsilon)M^2}\Big]= \tag{21}$$

$$=N\gamma_5\Big[-\frac{1}{-p^2+m^2}+\int_{1+2\sqrt{\varepsilon}}^{\infty}\frac{\mathrm{d}c}{c^2}\,\frac{1}{-p^2+m^2+cM^2}+0\,(\sqrt{\varepsilon})\Big]=$$

$$=N\gamma_5\,\frac{M^2}{(-p^2+m^2)^2}\ln\left(\frac{M^2+2mM}{M^2+2mM-p^2+m^2}\right)+0(\sqrt{\varepsilon})\,.$$

4. – Normalization of the B.S. wave function.

Since we intend to use the pseudoscalar solution (20) for computing the pion-nucleon coupling constant, it is of immediate importance to fix the correct normalization factor N. In order to do this we use the formula

$$\langle \pi | T\psi(x)\bar{\psi}(x') | \pi \rangle \simeq i \int \mathrm{d}z\, \mathrm{d}z' \langle \pi | T\psi(x)\bar{\psi}(z) | 0 \rangle \cdot \tag{22}$$

$$\cdot (i\gamma\partial_z + m) S_f(z - z') (i\gamma\partial_{z'} - m) \langle 0 | T\psi(z')\bar{\psi}(x') | \pi \rangle =$$

$$= i \int \mathrm{d}z\, \mathrm{d}z'\, \tilde{\chi}_q(x, z) S_f^{-1}(z - z') \chi_q(z', zx) .$$

Here

$$\tilde{\chi}_q(x, z) = \langle \pi | T\psi(x)\bar{\psi}(z) | 0 \rangle . \tag{23}$$

The above formula holds only up to terms that do not contain a one particle singularity ($\simeq$). It has been proved by ZIMMERMANN ([4]) who restricted himself to the scalar case, the generalization to the spinor case being obvious From (23), (6) by means of the reduction formula of Lehmann-Symanzik-Zimmermann one easily finds

$$\tilde{\chi}_q(x, z) = \chi_{-q}(x, z) = \exp\left[iq\,\frac{x+z}{2}\right] \varphi_{-q}(x - z) . \tag{24}$$

The expectation value of the current density may then be written ([5])

$$\langle \pi | j_0(x) | \pi \rangle = \langle \pi | T\bar{\psi}(x)\gamma_0\psi(x) | \pi \rangle = \mathrm{Sp}\,[\langle \pi | T\psi(x)\bar{\psi}(x) | \pi \rangle \gamma_0] \simeq \tag{25}$$

$$\simeq i \int \mathrm{d}z\, \mathrm{d}z'\, \mathrm{Sp}\,[\chi_{-q}(x, z) S_f^{-1}(z - z') \chi_q(z', x)\gamma_0] =$$

$$= i \int \mathrm{d}z\, \mathrm{d}z'\, \mathrm{d}p\, \mathrm{d}p'\, \mathrm{d}k \exp\left[i\,\frac{q}{2}(x+z) + ip(x-z) + ik(z-z') + ip'(z'-x) - i\,\frac{q}{2}(z'+x)\right] \cdot$$

$$\cdot (2\pi)^{-12} \mathrm{Sp}\,[\varphi_{-q}(p) S_f^{-1}(k) \varphi_q(p')\gamma_0] = i \int \frac{\mathrm{d}p}{(2\pi)^4}\, \mathrm{Sp}\,[\varphi_{-q}(p) S_f^{-1}(p - \tfrac{1}{2}q) \varphi_q(p)\gamma_0] .$$

The last expression of this relation exhibits the fact that $\langle \pi | j_4(x) | \pi \rangle$ does not depend on x. Since $\int \mathrm{d}^3x \langle \pi | j_4(x) | \pi \rangle$ has to be normalized to unity this

([4]) W. ZIMMERMANN: *Nuovo Cimento*, **13**, 503 (1959).

([5]) Strictly speaking, we can no longer neglect the isotopic degrees of freedom: we deal with $\langle \pi^+ | T\bar{\psi}_P(x)\gamma_0\psi_P(x) | \pi^+ \rangle$ and $\chi^+(x, x') = \langle 0 | T\psi_P(x)\bar{\psi}_N(x) | \pi^+ \rangle$.

means that

$$\langle \pi | j_4(x) | \pi \rangle = \frac{1}{V}, \tag{26}$$

where V is the volume of the normalization box. Inserting (25) into (26) we find

$$\frac{1}{V} = i \int \frac{\mathrm{d}p}{(2\pi)^4} \operatorname{Sp} \left[\varphi_{-q}(p) S_f^{-1}(p - \tfrac{1}{2}q) \varphi_q(p) \gamma_4 \right]. \tag{27}$$

Inserting (21) into (27) we obtain

$$\frac{1}{V} = iN^2 \int \frac{\mathrm{d}p}{(2\pi)^4} \operatorname{Sp} \left\{ \gamma_5 \left[\left(\gamma, p - \frac{q}{2} \right) - m \right] \gamma_5 \gamma_4 \right\} \cdot$$

$$\cdot \frac{(M^2)^2}{(-p^2 + m^2)^4} \ln^2 \left(\frac{M^2 + 2mM}{M^2 + 2mM - p^2 + m^2} \right) = \frac{N_4^2}{(2\pi)^4} \pi^2 \frac{q_0}{2} \int \mathrm{d}t\, t \frac{1}{(t + \varepsilon)^4} \cdot$$

$$\cdot \ln^2 (1 + t + \varepsilon) = \frac{N^2 q_0}{2(2\pi)^2} |\ln \varepsilon|,$$

up to terms of order unity. Hence

$$N = \frac{2\pi\sqrt{2}}{(q_0 V)^{\frac{1}{2}}} |\ln \varepsilon|^{-\frac{1}{2}}. \tag{28}$$

This relation shows that it is essential not to neglect the nucleon mass, since otherwise the B.S. wave function is not normalizable. It will be noticed that it is not consistent to neglect contributions to $\varphi(x)$ proportional to q when calculating N, as these will yield additional contributions of order q in (27). In a forthcoming paper we will take into account all the terms written down in eq. (8).

5. – The pion-nucleon coupling constant.

The renormalized pion-nucleon coupling constant is (6) the coefficient of γ_5 in the nucleon vertex function on the mass-shell which, since in our approximation $q = 0$, is

$$\Gamma(p, p)|_{p^2 = m^2} = ((\gamma p) - m) \varphi_q(p) ((\gamma p) - m) \sqrt{q_0 V}. \tag{29}$$

(6) J. V. LEPORE and K. M. WATSON: *Phys. Rev.*, **76**, 1157 (1949).

Inserting here (21) and taking account of the approximation which it implies we find

$$\Gamma(p, p)\,|_{p^2=m^2} =$$

$$= (\gamma p - m)\gamma_5(\gamma p - m)\sqrt{q_0 V}\,N \frac{M^2}{(-p^2 + m^2)^2} \cdot \ln\left(1 - \frac{p^2 - m^2}{M^2}\right)\Big|_{p^2=m^2} = \sqrt{q_0 V}\,N\,\gamma_5 .$$

Hence

$$g \simeq \sqrt{q_0 V}\,N = \frac{2\pi\sqrt{2}}{\sqrt{\ln M^2/m^2}}, \tag{30}$$

and a coupling constant of the order of unity results even for M that determine by (19) a value of λ corresponding to the coupling constant of the universal weak interaction, a result which supports the ideas expressed in the introduction to this paper. It should be remarked that by the Feynman cut-off procedure the $\ln^{-\frac{1}{2}}$ dependence of g has been deduced by MAKI (7).

(7) Z. MAKI: *Progr. Theor. Phys.*, **16**, 667 (1956).

RIASSUNTO (*)

Si propone di ridurre le interazioni forti alla interazione debole universale, considerandole manifestazioni ad alta energia dell'interazione debole fortemente dipendente dall'energia. Seguendo Fermi e Yang si suppone che il pione sia uno stato legato nucleone-antinucleone. Per risolvere l'equazione di Bethe-Salpeter si usa un metodo presentato in uno scritto precedente. Si normalizza la funzione d'onda dello stato legato e si calcola la costante di accoppiamento pione-nucleone. Quest'ultima risulta essere dell'ordine dell'unità quantunque si usi un'interazione debole fra i fermioni.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

A One-Dimensional Soluble Model of Quantum Field Theory with External Field.

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(ricevuto il 4 Agosto 1960)

Summary. — The one-dimensional model of interaction between the zero mass quantized Dirac field with a given time-dependent external field is solved exactly. The field operator ψ is given as a functional of the corresponding incoming field operator φ. The S operator is brought into the ordered form which gives the possibility of investigating any process for any strength of coupling. Finally the vacuum-to-vacuum transition is compared with the results given by M. Neuman, A. Salam and P. T. Matthews.

1. – Introduction.

In this paper we shall study the one-dimensional model of the interaction between the zero mass quantized Dirac field with a given time-dependent external field. We shall be able to give a complete solution of this problem in a closed form.

The form of the interaction is the same as of the one-dimensional quantum electrodynamics with zero mass electrons, which was studied by V. Glaser and B. Jakšić ([1]). They have shown that the model of one-dimensional interaction between zero mass electrons with photons is essentially equivalent to the Thirring model ([2]), which was solved exactly by V. Glaser ([3]).

([1]) V. Glaser and B. Jakšić: *Nuovo Cimento*, **11**, 877 (1959).
([2]) W. E. Thirring: *Ann. Phys.*, **3**, 91 (1958); *Nuovo Cimento*, **9**, 1007 (1958).
([3]) V. Glaser: *Nuovo Cimento*, **9**, 990 (1958).

In the three-dimensional space the problem of the quantized Dirac field in interaction with a given external field, which is time-dependent, was studied by A. SALAM, P. T. MATTHEWS (4), R. P. FEYNMAN (5) and others by the use of perturbation theory. A. SALAM and P. T. MATTHEWS have shown that for an external field for which the Fredholm theory can be used, the probability amplitude exists for any value of the coupling constant. For instance, the real part of the probability amplitude for scattering was given in principle as the Fredholm resolvent, which is an infinite absolutely convergent series in the coupling constant.

In case the three-dimensional space is reduced to one dimension, and zero mass electrons are taken we can obtain a probability amplitude for any process as an analytical expression of the coupling constant. This model is only an example of the quantum field theory which can be solved exactly.

In the second section are given the solutions for the field operators and the formulation of the S operator. The third section is devoted to the reduction of the S operator into the ordered form. In the fourth section some processes are considered, and the last section contains a comparison of the vacuum-expectation value with the results obtained by several authors. In the Appendix we give the integral transformation operators used in solving a singular integral equation.

2. – Solution of the equation of motion; the formulation of the S operator.

Let us start with the equations of the field operators

$$-i\gamma^\mu \partial_\mu \psi(x,t) = g A_\mu(x,t)\gamma^\mu \psi(x,t)\,, \tag{1}$$

where $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, $g^{11} = -g^{22} = 1$, $g^{12} = g^{21} = 0$ and

$$\{\psi^*_\alpha(x,t), \psi_\beta(x',t)\} = \delta_{\alpha\beta}\delta(x-x')\,, \qquad \{\psi_\alpha(x,t), \psi_\beta(x',t)\} = 0\,, \tag{2}$$

α and $\beta = 1, 2$ represent the spinor indices.

For the Dirac matrices we can take the representation according to THIRRING (2):

$$\gamma^1 = i\sigma_1\,, \qquad \gamma^2 = \sigma_2\,, \tag{3}$$

where σ_r are the Pauli matrices.

(4) A. SALAM and P. T. MATTHEWS: *Phys. Rev.*, **90**, 690 (1953).
(5) R. P. FEYNMAN: *Phys. Rev.*, **75**, 486, 1736 (1949).

Using the representation (3) and introducing new variables $u = x + t$, $v = x - t$, we can write the equation (1) in the following form:

$$(1') \qquad \begin{cases} \dfrac{\partial}{\partial u}\psi_1 = ig\overline{A}_1\psi_1 , \\ \dfrac{\partial}{\partial v}\psi_2 = -ig\overline{A}_2\psi_2 , \end{cases}$$

where

$$(4) \qquad \overline{A}_1 = \tfrac{1}{2}(A_1 + A_2) , \qquad \overline{A}_2 = \tfrac{1}{2}(A_2 - A_1) .$$

The general solution of the system (1′) is

$$(5) \qquad \begin{cases} \psi_1 = \varphi_1(v) \exp\left[ig\int_{-\infty}^{u}\overline{A}_1(u, v)\,\mathrm{d}u\right] , \\ \psi_2 = \varphi_2(u) \exp\left[ig\int_{v}^{+\infty}\overline{A}_2(u, v)\,\mathrm{d}v\right] , \end{cases}$$

where φ_1 and φ_2 are the interaction-free solutions of (1′). For $t \to -\infty$ ($u \to -\infty$, $v \to +\infty$) $\psi_{1,2}$ behave asymptotically as the interaction-free operators $\varphi_{1,2}$ (see [3]). The interaction-free operators $\varphi_{1,2}$ satisfy the same anti-commutation relations as $\psi_{1,2}$. Moreover, in the new variables u, v, the free operators $\varphi_{1,2}$ satisfy the following anti-commutation relations [3]

$$(6) \qquad \{\varphi_\alpha(y), \varphi^*_\beta(y')\} = \delta_{\alpha\beta}\,\delta(y - y') , \qquad \{\varphi_\alpha(y), \varphi_\beta(y')\} = 0 ,$$

where y represents v respectively u for $\alpha = 1, 2$.

We shall need the form of the Fourier representation as given in [3]

$$(7) \qquad \begin{cases} \varphi_\alpha(y) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} c_\alpha(p) \exp\,[ipy]\,\mathrm{d}p , \\ \{c_\alpha(p), c^*_\beta(p')\} = \delta_{\alpha\beta}\,\delta(p - p') , \ \{c_\alpha(p), c_\beta(p')\} = 0 , \end{cases}$$

and the standard corresponding spinor form

$$(8) \qquad \varphi_{1,2}(x, t) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty}\theta_\pm(p)\,\{a(p) \exp\,[i(px - |p|t)] + b^*(p) \exp\,[-i(px - |p|t)]\} ,$$

where

$$\theta_+(p) = \begin{cases} 1\,, & p > 0\,, \\ 0\,, & p < 0\,, \end{cases} \qquad \theta_-(p) = \theta_+(-p)\,.$$

$a(p)$ and $b(p)$ are the annihilation operators of an electron and a positron, respectively.

The connection between the operators $c_\alpha(p)$ and the operators $a(p)$ and $b(p)$ is as follows:

$$(9) \qquad \begin{cases} c_{1,2}(p) = \theta_\pm(p)\, a(p) + \theta_\mp(p)\, b^*(-p)\,, \\ \{a(p),\, a^*(p')\} = \{b(p),\, b^*(p')\} = \delta(p - p')\,, \\ \{a(p),\, a(p')\} = \{b(p),\, b(p')\} = 0\,. \end{cases}$$

The physical vacuum and the states of n free electrons and m free positrons are defined by

$$(10) \qquad \begin{cases} a(p)\,|0\rangle = b(p)\,|0\rangle = 0\,, \\ |p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_m\rangle = \dfrac{1}{\sqrt{n!\,m!}}\, a^*(p_1) \ldots a^*(p_n) b^*(q_1) \ldots b^*(q_m)\,|0\rangle\,. \end{cases}$$

From (7) and (9) we see that φ_α can be decomposed into the positive and negative frequency parts

$$\varphi_\alpha = \varphi_\alpha^+ + \varphi_\alpha^-\,,$$

which satisfy the following anti-commutation rules

$$(6') \qquad \{\varphi_\alpha^\pm(y),\, \varphi_\beta^{\pm *}(y')\} = \delta_{\alpha\beta}\,\delta_\pm(y - y')\,.$$

Here

$$\delta_\pm(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \theta_\pm(p)\, \exp\,[ipx]\, \mathrm{d}p\,.$$

The contractions of free operators in an ordinary product will be

$$(11) \qquad \begin{cases} \varphi_\alpha(y)\,\varphi_\beta^*(y') = \varphi_\alpha^*(y)\,\varphi_\beta(y') = \delta_{\alpha\beta}\,\delta_\alpha(y - y')\,, \\ \varphi_\alpha\varphi_\beta = \varphi_\alpha^*\varphi_\beta^* = 0\,, \qquad \delta_{1,2}(y - y') = \delta_\pm(y - y')\,. \end{cases}$$

If we define

$$\begin{cases} \varrho_1(v) = \psi_1^*(x,t)\,\psi_1(x,t) = \varphi_1^*(v)\,\varphi_1(v)\,, \\ \varrho_2(u) = \psi_2^*(x,t)\,\psi_2(x,t) = \varphi_2^*(u)\,\varphi_2(u)\,, \end{cases} \tag{12}$$

we can show, by using the anti-commutators (6) for evaluating the commutators between the ϱ's, that the following relations are valid:

$$\begin{cases} [\varrho_1, \varrho_2] = 0\,, \\ [\varrho_1(y), \varrho_1(y')] = \dfrac{i}{\pi}\dfrac{P}{y-y'}\,\delta(y-y')\,, \\ [\varrho_2(y), \varrho_2(y')] = -\dfrac{i}{\pi}\dfrac{P}{y-y'}\,\delta(y-y')\,, \end{cases} \tag{13}$$

where we are allowed to put

$$\delta(y-y')[:\varphi_\alpha^*(y)\,\varphi_\alpha(y'):-:\varphi_\alpha^*(y')\,\varphi_\alpha(y):] = 0\,, \tag{13'}$$

because all matrix elements of that operator between the physical states defined in (10) vanish. The last two commutators in (13) are not well defined, since they are proportional to the undefined product of distributions $P/(y-y')\cdot\delta(y-y')$. We shall show that the physical results are independent of the definition of this product.

The interaction Lagrangian density of our model (see also [1])

$$\mathscr{L}_1(x,t) = g j_\mu(x,t) A^\mu(x,t)\,, \qquad j^\mu = \bar{\psi}\gamma^\mu\psi = (\varrho_1 - \varrho_2,\ \varrho_1 + \varrho_2)$$

can be written by using the definition (4) in the following form

$$\mathscr{L}_1 = 2g(\varrho_1 \overline{A}_1 + \varrho_2 \overline{A}_2)\,.$$

With the help of (13) it is easy to find that the commutator between the total interaction Lagrangian $L_1(t) = \int \mathscr{L}_1(x,t)\,\mathrm{d}x$ at two different times is as follows:

$$[L_1(t), L_1(t')] = i\,C(t,t')\,, \tag{14}$$

where $C(t,t')$ is an antisymmetric real c-number function, because g and A^μ are real.

The function $C(t,t')$ in (14) is not determined, since it is given as a combination of space integrals over the distribution product found in (14).

The S operator can be found exactly, regardless of the choice for $C(t,t')$,

by the method for treating the infrared catastrophe (see ref. (6)). By applying this method for determining the S operator, the indeterminate function $C(t, t')$ will appear in the phase factor in the following form

$$\exp\left[\frac{i}{2}\int_{-\infty}^{+\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t'\,C(t', t)\right].$$

It is clear that such a phase factor can be omitted in the definition of the S operator, because it does not change the physical process.

It is convenient, however, to replace $\varrho_\alpha \to :\varrho_\alpha:$ so that finally for the S operator we obtain

$$S = \exp\left[2ig\int_{-\infty}^{+\infty}\mathrm{d}x\,\mathrm{d}t(:\varrho_1:\bar{A}_1 + :\varrho_2:\bar{A}_2)\right],$$

where it can easily be seen that the operators $:\varrho_\alpha:$ still fulfil the commutation relations (13). After introducing $u = x+t$, $v = x-t$, according to the relations (4) we can finally rewrite the S operator in the following form

$$(15)\qquad \begin{cases} S = \exp\left[ig\int_{-\infty}^{+\infty}\mathrm{d}y\sum_{\alpha=1}^{2} a_\alpha(y):\varrho_\alpha(y):\right], \\ a_1(v) = \int_{\infty}^{+\infty}\bar{A}_1(u, v)\,\mathrm{d}u\ , \qquad a_2(u) = \int_{-\infty}^{+\infty}\bar{A}_2(u, v)\,\mathrm{d}v\ , \ (*) \end{cases}$$

where u and v are replaced by y under the sign of integration.

The S operator can be represented as a product of two independent parts. This fact results from the first of the commutation relations (13), so that we have

$$(16)\qquad \begin{cases} S = S_1 \cdot S_2\ , \qquad [S_1, S_2] = 0\ , \\ S_\alpha = \exp\left[ig\int_{-\infty}^{+\infty}\mathrm{d}y\,a_\alpha(y):\varrho_\alpha(y):\right], \qquad \alpha = 1, 2. \end{cases}$$

This fact will help us in the calculation to be performed in the next section.

(6) J. M. Jauch and F. Rohrlich: *Helv. Phys. Acta*, **27**, 613 (1954).

(*) Because A_μ vanishes in time and space infinity, $a_\alpha(\infty)=0$, $\alpha=1, 2$.

3. – The ordered form of the S operator.

As it is known from perturbation theory we have to decompose each term of the S operator into ordered products. This procedure enables us to study any process of approximation in powers of the coupling constant. In our case we shall be able to get the exact S operator in the ordered form, which makes it possible to obtain the probability amplitude for all processes as an analytical expression of the coupling constant. The expression thus obtained will be valid for any value of the coupling constant.

Following Glaser's procedure of reducing the exponential operator $\exp[iQ]$, where Q is bilinear in free field operators, into an ordered form, we shall start from the differential equation for S_α obtained from (16) (*),

$$\frac{1}{i}\frac{\partial}{\partial g} S_\alpha = \int_{-\infty}^{+\infty} :\varrho_\alpha(y): a_\alpha(y)\,\mathrm{d}y\, S_\alpha \,. \tag{17}$$

The right-hand side of (17) can be brought into the form

$$\frac{1}{i}\frac{\partial}{\partial g} S_\alpha = \int_{-\infty}^{+\infty} \mathrm{d}y\,\mathrm{d}y' A_\alpha(y, y'; g);\ \varphi_\alpha^*(y)\varphi_\alpha(y') S_\alpha;\ + f_\alpha(g) S_\alpha \,. \tag{18}$$

In the equation (18) the expression $;\varphi_\alpha^*(y)\varphi_\alpha(y') S_\alpha;$ is in the semiordered form. *i.e.* all the creation operators stand to the left, and all the annihilation operators stand to the right of S_α. The operators a, b, a^*, b^* between semicolons are all anti-commuting between themselves and commuting with S_α (**). The c-number functions $A_\alpha(y', y; g)$ and $f_\alpha(g)$ appear as the result of semiordering the right-hand side of the equation (17). The solution of (18) can immediately be written down in the following form:

$$S_\alpha = \langle S_\alpha \rangle_0 :\exp\left[\int \mathrm{d}y\,\mathrm{d}y' M_\alpha(y, y'; g)\varphi_\alpha^*(y)\varphi_\alpha(y')\right], \tag{19}$$

where $M_\alpha(y, y'; g)$ is defined as

$$M_\alpha(y, y'; g) = i\int_0^g A_\alpha(y, y'; g)\,\mathrm{d}g \,,$$

(*) See formulas (34)-(41) ref. [3].
(**) For details see ref. [3].

the vacuum expectation value $\langle S_\alpha \rangle_0$ is defined as follows:

$$\langle S_\alpha \rangle_0 = \exp\left[i \int_0^g f_\alpha(g)\, \mathrm{d}g \right]. \tag{20}$$

To obtain the functions $M_\alpha(y, y'; g)$ and $f_\alpha(g)$ which according to (19) uniquely determine the ordered form of the S operator, we shall start with the fact that

$$\varphi_\alpha(y) S_\alpha = \exp\left[ig\, a_\alpha(y)\right] S_\alpha \varphi_\alpha(y)\,. \tag{21}$$

If we take $\alpha = 1$ in (21) and apply the integral operator H^- (see the Appendix) to the relation (21) then after a simple recombination we shall obtain

$$;S_1\varphi_1; = S_1\varphi_1 + H^-\{(\exp\left[iga_1\right] - 1) S_1\varphi_1\}\,.$$

To express $S_1\varphi_1$ in terms of $;S_1\varphi_1;$ we have to solve this integral equation. For the case $\alpha = 2$, we could obtain a similar expression, so that we have for both cases

$$\begin{cases} ;S_\alpha\varphi_\alpha; = \{1 + K_\alpha\} S_\alpha\varphi_\alpha, \\ K_1(y, y'; g) = \delta_-(y - y')(\exp\left[ig\, a_1(y')\right] - 1)\,, \\ K_2(y, y'; g) = \delta_+(y - y')(\exp\left[ig\, a_2(y')\right] - 1)\,. \end{cases} \tag{22}$$

The integral equations in (22) can be transformed into the singular integral equation (see (A.4)), so that after a simple calculation its solution (A.9) gives us $S_\alpha\varphi_\alpha$ in terms of $;S_\alpha\varphi_\alpha;$ uniquely in the following way

$$\begin{cases} S_\alpha\varphi_\alpha = \{1 + G_\alpha\};S_\alpha\varphi_\alpha; \\ G_1(y, y'; g) = \exp\left[-ig\, a_1^-(y)\right] \delta_-(y - y') \exp\left[-ig\, a_1^+(y')\right] \cdot \\ \qquad\qquad \cdot\left(1 - \exp\left[ig\, a_1(y')\right]\right), \\ G_2(y, y'; g) = \exp\left[-ig\, a_2^+(y)\right] \delta_+(y - y') \exp\left[-ig\, a_2^-(y')\right] \cdot \\ \qquad\qquad \cdot\left(1 - \exp\left[ig\, a_2(y')\right]\right), \\ a_\alpha^\pm(y) = H_y^\pm\left[a_\alpha(y')\right]. \end{cases} \tag{23}$$

The following relation exists between the integral operators K and G, by which (23) is uniquely determined:

$$K_\alpha G_\alpha = G_\alpha K_\alpha = -(K_\alpha + G_\alpha)\,. \tag{24}$$

By the first of the relations (23) and by using (21) we can get

$$(25)\qquad \begin{cases} [S_\alpha, \varphi_\alpha] = (1 - \exp[ig a_\alpha])[1 + G_\alpha]\,; S_\alpha \varphi_\alpha; \\ [\varphi_\alpha^*, S_\alpha] = (1 - \exp[ig a_\alpha])[1 + \bar{G}_\alpha]\,; S_\alpha \varphi_\alpha^*; \end{cases}$$

where $\bar{G}_\alpha(g) = G_\alpha^*(-g)$.

All other commutators, as for example $[S_\alpha, q_\alpha^\pm]$, can be found by applying $H^\pm$ to the relations (25) and by using (24). Now we see that the equation (18) is easy to obtain, by applying the relations (25) to the relation (17). The solution for the operator S_α in the form (19) is possible.

In order to obtain the function $M_\alpha(y, y'\,; g)$ we shall calculate the vacuum expectation value of $\varphi_\alpha^*(y)\,\varphi_\alpha(y') S_\alpha$. Using the relations (6'), (23) and (25) we finally obtain

$$(26)\qquad \langle \varphi_\lambda^*(y) \varphi_\lambda(y') S_\alpha \rangle_0 = \exp[ig a_\lambda(y')] \left[\delta_\lambda(y - y') + \int G_\lambda(y', y'') \delta_\lambda(y - y'')\, \mathrm{d}y'' \right] \langle S_\alpha \rangle_0 .$$

Let us consider the identity

$$(27)\qquad \langle \{\varphi_\alpha^*(y), [\varphi_\alpha(y'), S_\alpha]\} \rangle_0 = (\exp[ig a_\alpha(y')] - 1) \langle \{\varphi_\alpha^*(y), S_\alpha \varphi_\alpha(y')\} \rangle_0 .$$

The right-hand side of (27) can be calculated by using (26). The solution of S_λ in the form (19) will be introduced into the left-hand side, so that after calculation we compare both sides of (27) and obtain M_α in the following form:

$$(28)\qquad M_\alpha(y, y'\,; g) = (\exp[ig a_\alpha(y)] - 1)[\delta(y - y') + G_\alpha(y, y'\,; g)] .$$

To obtain the vacuum expectation values $\langle S_\alpha \rangle_0$ and the function $f_\alpha(g)$ respectively, we shall consider the vacuum expectation value of the differential equation (17)

$$(29)\qquad \frac{1}{i} \frac{\partial}{\partial g} \langle S_\alpha \rangle_0 = \int_{-\infty}^{+\infty} \mathrm{d}y\, a_\alpha(y) \langle : \varrho_\alpha(y) : S_\alpha \rangle_0 .$$

If we put the expression (19) for S_λ into the right-hand side of the equation (29) then after the evaluation of the matrix element according to the relation (20) we shall be able to find $f_\alpha(g)$ in the form

$$(30)\qquad \begin{cases} f_\alpha(g) = \int \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, a_\lambda(x)\, M_\alpha(y, z\,; g)\, \delta_\alpha(x - y)\, \delta_\alpha(x - z) = \\ \qquad\qquad = \int \mathrm{d}p\, \mathrm{d}q\, \theta_\alpha(p) \theta_\alpha(q)\, \tilde{a}_\alpha(-p - q)\, \tilde{M}_\alpha(p, q\,; g) , \\ \theta_1(p) = \theta_+(p) , \qquad \theta_2(p) = \theta_-(p) , \end{cases}$$

where we introduce the following Fourier representations

$$(31)\qquad \begin{cases} a_\alpha(y) = \int \tilde{a}_\alpha(p) \exp\,[ipy]\,\mathrm{d}p\,, \\ \tilde{a}_\alpha(p) = \dfrac{1}{2\pi}\int a_\alpha(y) \exp\,[-ipy]\,\mathrm{d}y\,, \end{cases}$$

$$(32)\qquad \begin{cases} M_\alpha(y, y'\,; g) = \dfrac{1}{2\pi}\int \tilde{M}_\alpha(p, q\,; g) \exp\,[i(py + qy')]\,\mathrm{d}p\,\mathrm{d}q\,, \\ \tilde{M}_\alpha(p, q\,; g) = \dfrac{1}{2\pi}\int M_\alpha(p, q\,; g) \exp\,[-i(py + qy')]\,\mathrm{d}y\,\mathrm{d}y'\,. \end{cases}$$

4. – The probability amplitude for scattering and pair creation.

To study the probability amplitude of these processes we shall consider the final state to be always different from the initial one. It will be convenient to use the S operator in the momentum representation, so that according to the relations (7), (16) and (32) we get

$$(33)\qquad \begin{cases} S = \langle S\rangle_0 {:}R_1{:}\,{:}R_2{:} = \langle S\rangle_0 {:}R_1 R_2{:}\,, \\ R_\alpha = \exp\left[\int \mathrm{d}p\,\mathrm{d}q\,\tilde{M}_\alpha(p, -q\,; g)\, c^*_\alpha(p)\, c_\alpha(q)\right], \\ \langle S\rangle_0 = \langle S_1\rangle_0 \langle S_2\rangle_0\,. \end{cases}$$

In the following calculations we shall frequently need the anti-commutators:

$$(34)\qquad \begin{cases} \{c_\alpha(p'),\, a^*(p)\} = \theta_\alpha(p)\,\delta(p - p')\,, \\ \{c^*_\alpha(p'),\, b^*(p)\} = \theta_\alpha(p)\,\delta(p + p')\,, \\ \{c_\alpha(p'),\, a\ (p)\} = \{c_\alpha(p'),\, b^*(p)\} = 0\,. \end{cases}$$

First we shall consider the scattering of an electron on the field A_μ from the initial state $|p_i\rangle$ to the final state $|p_f\rangle$. The amplitude of the transition probability is written as $\langle p_f|S|p_i\rangle$. To calculate that matrix element we use (33) and (34) and obtain

$$(35)\qquad \langle p_f|S|p_i\rangle = \langle S\rangle_0\{\tilde{M}_1(p_f, -p_i\,; g)\,\theta_+(p_i)\,\theta_+(p_f) + \tilde{M}_2(p_f, -p_i\,; g)\,\theta_-(p_i)\,\theta_-(p_f)\}\,.$$

The transition probability can be obtained as an absolute square of the matrix element (35). It is clear that the transition probability is given as an analytical expression which is valid for any value of g, *i.e.* for any strength of field.

In the one-dimensional space both the initial and the final momenta of an electron have the same directions.

In order to obtain the matrix element of the probability amplitude for the creation of any number of pairs, we shall redefine the state vector in (10) for n-pairs of electron-positrons so that

$$|p_1, q_1; \ldots; p_n, q_n\rangle = \frac{1}{n!}\, a^*(p)_1 b^*(q_1) \ldots a^*(p_n) b^*(q_n)|0\rangle \,. \tag{36}$$

The expression (36) differs only by an irrelevant phase factor $(-)^{n(n-1)/2}$ from the vector which follows from the definition (10). We are able to obtain the matrix element $\langle p_1, q_1; \ldots; p_n, q_n|S|0\rangle$, by using the anticommutation relations (34). After a straighforward calculation we obtain

$$\langle p_1, q_1; \ldots; p_n, q_n|S|0\rangle = \tag{37}$$

$$= \frac{\langle S\rangle_0}{n!}\left\{\frac{1}{n!}\sum_{\alpha=1}^{2}\sum_{\varrho,\sigma}(-)^{\varrho+\sigma}\theta_\alpha(p_{\varrho_1})\ldots\theta_\alpha(p_{\varrho_n})\theta_\alpha(q_{\sigma_1})\ldots\theta_\alpha(q_{\sigma_n})\widetilde{M}_\alpha(p_{\varrho_1}, q_{\sigma_1})\ldots\widetilde{M}_\alpha(p_{\varrho_n}, q_{\sigma_n}) + \right.$$

$$+\sum_{k=1}^{n-1}\frac{1}{k!\,(n-k)!}\sum_{\varrho,\sigma}(-)^{\varrho+\sigma}\theta_+(p_{\varrho_1})\ldots\theta_+(p_{\varrho_k})\theta_-(p_{\varrho_{k+1}})\ldots\theta_-(p_{\varrho_n})\theta_+(q_{\sigma_1})\ldots\cdot$$

$$\left.\cdot\ldots\theta_+(q_{\sigma_k})\theta_-(q_{\sigma_{k+1}})\ldots\theta_-(q_{\sigma_n})\widetilde{M}_1(p_{\varrho_1}, q_{\sigma_1})\ldots\widetilde{M}_1(p_{\varrho_k}, q_{\sigma_k})\,\widetilde{M}_2(p_{\varrho_{k+1}}, q_{\sigma_{k+1}})\ldots\widetilde{M}_2(p_{\varrho_n}, q_{\sigma_n})\right\},$$

where the summation over the indices ϱ_i and σ_i means the summation over all possible permutations from 1 to n. The starting arrangement of the indices is $1 \ldots n$ for each k.

From (37) we see that the emission of pairs electron-positron is possible in both directions in any combination in which they occur, but in such a way that the charge is conserved in each direction separately.

For instance, one pair has two possibilities, either to move to the left or to the right after creation. One would perhaps expect Poisson's distribution in the one-dimensional space from the vacuum-to-n-pairs transition, but Poisson's distribution does not occur in this case, which one can show by direct calculations. It seems that Poisson's distribution is bound only by the Bose statistics but, of course, we are not able to give any answer without more detailed study.

5. – The connection with perturbation theory.

According to perturbation theory the processes of this model can be studied by making the corresponding calculation with the S operator in the exponential form (16) developed in powers of the coupling constant g.

In this section we shall study only the vacuum-to-vacuum transition. For that transition there are some calculations in perturbation theory given by several authors, and we are only interested to see whether our result for $\langle S \rangle_0$ is in agreement with theirs.

To do that, we shall first consider some properties of the function $f_\alpha(g)$, which determine the vacuum-expectation value (see (20) and (30)). From (30) one can see that the whole dependence on g results from the function $M_\alpha(y, z\ ; g)$, therefore we shall show an important property of this function. By using the properties of $\delta_\pm$ functions and the operators $H^\pm$, we can prove that

$$M_\alpha(y, z\ ; -g) = -M_\alpha(z, y\ ; g)\,, \tag{38}$$

i.e. the function $f_\alpha(g)$ is anti-symmetric with respect to the variable g

$$f_\alpha(-g) = -f_\alpha(g)\,. \tag{39}$$

According to the relations (39) and (20) we are able to rewrite the vacuum expectation value $\langle S_\alpha \rangle_0$ in perturbation theory

$$\langle S_\alpha \rangle_0 = \exp\left[-\sum_{n=1}^{\infty} \frac{\sigma_{2n}(\alpha) g^{2n}}{2n}\right], \tag{40}$$

where σ_{2n}'s are the negative coefficients of the developed function $if_\alpha(g)$.

The result for $\langle S_\alpha \rangle_0$ is in agreement with the form given by M. NEUMANN (7), A. SALAM and P. T. MATTHEWS (4). The result (40) is clear if we take into account that in our model

$$\langle : \varrho_\alpha(y_1) : \ldots : \varrho_\alpha(y_{2n+1}) : \rangle_0 = 0$$

for n integer.

* * *

The author wishes to thank Dr. B. JAKŠIĆ and Dr. V. GLASER for many helpful discussions and the great interest in the course of this work.

APPENDIX

The integral operators $H^\pm$ are defined as

$$H_y^\pm[f(y')] = \int_{-\infty}^{+\infty} \delta_\pm(y - y') f(y')\ \mathrm{d}y'\,,$$

(7) M. NEUMAN: *Phys. Rev.*, **83**, 1258 (1952); **85**, 129 (1952).

where

$$\delta_{\pm}(y-y') = \pm\,(2\pi i)^{-1}(y'-y_{\pm})^{-1}, \qquad y^{\pm} = y \pm i\varepsilon\,, \qquad \varepsilon \to +0\,.$$

The connection with the usual Hilbert integral operator H,

$$H_y[f(y')] = \pi^{-1}\int_{-\infty}^{+\infty} P[(y'-y)^{-1}]f(y')\,dy', \qquad H^2 = -1\,,$$

is given as

$$H^{\pm} = \tfrac{1}{2}(1 \mp iH)\,, \tag{A.1}$$

because

$$\delta_{\pm}(y) = \frac{1}{2}\,\delta(y) \mp \frac{1}{2\pi i}\,\frac{P}{y}\,.$$

From (A.1) we can verify some simple properties for $H^{\pm}$

$$\begin{cases} (H^{\pm})^2 = H^{\pm}, & H^{\pm}H^{\mp} = 0\,, \\ H^{+} + H^{-} = 1, & H^{+} - H^{-} = -iH\,. \end{cases} \tag{A.2}$$

The relation (A.2) means that the integral operator $H^{\pm}$ is a projection operator, *i.e.* we can decompose a function $f(y)$ into two parts

$$f(y) = f^{+}(y) + f^{-}(y)\,, \qquad f^{\pm}(y) = H_y^{\pm}[f(y')]\,, \tag{A.3}$$

where it is easy to see that $f^{\pm}$ is represented in the momentum-space by the positive or negative frequency.

In order to solve the integral equation

$$a(y)h(y) + i\lambda(y)H_y[h(y')] = g(y)\,, \tag{A.4}$$

$$-\infty < y < +\infty\,, \qquad h(\infty) \quad \text{and} \quad g(\infty) \to 0\,,$$

according to (A.2) we shall rewrite it in the form

$$[a(y) - \lambda(y)]h^{+}(y) + [a(y) + \lambda(y)]h^{-}(y) = g(y)\,. \tag{A.5}$$

If we put

$$h^{\pm}(y) = \exp\,[\pm\,\alpha^{\pm}(y)]\cdot\beta^{\pm}(y)\,, \tag{A.6}$$

we shall see that it is convenient to make the following requirement:

$$[a(y) - \lambda(y)]\exp\,[\alpha^{+}(y)] - [a(y) + \lambda(y)]\exp\,[-\alpha^{-}(y)] = 0\,, \tag{A.7}$$

from which we find that

$$\alpha(y) = \ln \frac{a(y) + \lambda(y)}{a(y) - \lambda(y)} .$$

After that it is easy to find $\beta(y)$, which is given as

(A.8) $$\beta(y) = g(y)[a(y) - \lambda(y)]^{-1} \exp[-\alpha^{+}(y)] = g(y)[a(y)+\lambda(y)]^{-1} \exp[\alpha^{-}(y)].$$

The solution for $h(y)$ (see (A.3.)) is finally given

(A.9) $$h(y) = \exp[\alpha^{+}(y)] H_y^{+} \left[\frac{g(y') \exp[\alpha^{-}(y')]}{a(y') + \lambda(y')}\right] + \\ + \exp[-\alpha^{-}(y)] H_y^{-} \left[\frac{g(y') \exp[-\alpha^{+}(y')]}{a(y') - \lambda(y')}\right].$$

RIASSUNTO (*)

Si risolve con esattezza il modello unidimensionale della interazione fra il campo quantizzato di Dirac della massa zero con un dato campo esterno dipendente dal tempo. L'operatore di campo ψ viene dato come funzionale del corrispondente operatore di campo entrante φ. L'operatore S è ridotto nella forma ordinata che dà la possibilità di analizzare qualsiasi processo qualunque sia la forza dell'accoppiamento. Infine si confronta la transizione vuoto-vuoto con i risultati dati da M. Neuman, A. Salam e P. T. Matthews.

(*) *Traduzione a cura della Redazione.*

Some Rare Decay Modes of the K-Meson.

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(ricevuto il 5 Agosto 1960)

Summary. — In view of future experiments with high intensity K-meson beams the branching ratios for $K^{\pm} \to \pi^{\pm} + e^+ + e^-$ and $K_1^0 \to 2\gamma$ are evaluated.

1. – The expected availability of high intensity K-meson beams makes it necessary to evaluate the branching ratio to be expected for presumably very rare decay modes of the K-mesons. In the first part of this note we consider the possible decay mode

$$K^{\pm} \to \pi^{\pm} + e^+ + e^- \tag{1}$$

due to the virtual step $K^{\pm} \to \pi^{\pm} + \gamma$ and $\gamma \to e^+ + e^-$. The process is of the second order in the e.m. coupling constant, but it is expected to have a fairly low branching ratio in view of the fact that for a spin zero K-meson the first of the two virtual steps is forbidden when the photon is transverse and on the energy shell. In the second part of the paper we consider the decay modes $K^0 \to 2\gamma$. In discussing such reactions one has to analyze first the consequences of the application of the *CP* selection rules to the K^0-decays (1).

(1) L. Landau: *Nucl. Phys.*, **3**, 127 (1957); R. Gatto: *Phys. Rev.*, **106**, 168 (1957). T. D. Lee and C. N. Yang: *Elementary particles and weak interactions*, B.N.L. 443 (1957).

From *CP* invariance the matrix elements for the 2γ decay of the shortlived K^0, K_S^0, is of the form $(\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2)$ where $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$ are the polarization vectors of the photons, while the matrix element for the longlived K^0, K_L^0, is of the form $\boldsymbol{q}(\boldsymbol{\epsilon}_1 \wedge \boldsymbol{\epsilon}_2)$, where $\boldsymbol{q}$ is the relative final momentum. The lowest mass intermediate states for K_S^0 is the 2π state, while for K_L^0 it is the 3π state. Our calculation concerns K_S^0 via the lowest mass intermediate state, but we expect a similar branching ratio for K_L^0 considering that its effective coupling is smaller and also its lifetime is longer.

2. – The simplest diagram which represents reaction (1) (let us choose the case of a K^+) is the following one, involving a vertex $(K^+\pi^+\gamma)$.

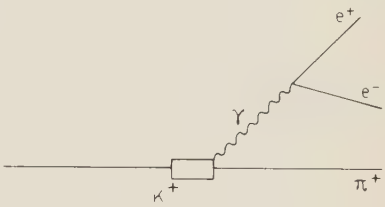

Fig. 1.

The vertex $(K^+\pi^+\gamma)$ must vanish for real photons because of gauge invariance.

The matrix element for process (2) will then be ($q = k - p$, where k and p are the 4-momenta of the K^+ and the pion, and p^+, p^- the momenta of the two electrons)

$$(2) \qquad \langle pp^+p^- \,|\, k\rangle = \frac{(2\pi)^{-3}}{q^2} \langle p \,|\, J_\mu \,|\, k\rangle \, \delta^4(k - p - p^+ - p^-)[u(p^-)\gamma_\mu v(p^+)] .$$

We make the position $\langle p \,|\, J_\mu \,|\, k\rangle = (2\pi)^{-3}(4E_pE_K)^{-\frac{1}{2}}C_\mu$, where $(p-k)_\mu C_\mu = 0$ due to gauge invariance. This condition fixes C_μ to be proportional to $\{k_\mu[\mu^2 + (pk)] + p_\mu[m_K^2 + (pk)]\}$. We evaluate the decay rate by performing first a summation over the spins and momenta of the electrons and we get

$$\text{Rate}\,(K^+ \to \pi^+ + e^+ + e^-) = \frac{e^2}{12 m_K (2\pi)^{12}} \int \frac{d^3p (C_\mu C_\mu)}{E_\nu[-(p-k)^2]} .$$

We shall now try to evaluate C_μ.

We can describe the $(K^+\pi^+\gamma)$ vertex in the simplest way by means of an intermediate $\bar{\Lambda}$-p loop, obtaining the following diagrams:

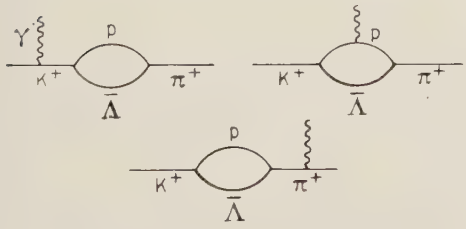

Fig. 2.

The left hand side vertex in the diagrams gives substantially the $(K^+p\Lambda)$ coupling constant, while the right hand side vertex is related to the lifetime of the Λ^0-hyperon. Of course in a better calculation the effect of intermediate Σ's should also be included.

One finds that the dominant

part of the contribution of the above graphs (*i.e.* the contribution obtained by letting the masses of the intermediate fermions go to infinity) vanishes identically, as a consequence of a general theorem stated by CABIBBO and GATTO which can be easily extended to cover this case (2).

The consequence is that the contribution of the above graphs is of the order $(m/M)^2$, where m is of the order of magnitude of the masses of the external particles, and M of the particles of the loop. The integrals are now convergent. A detailed calculation to the order $(m/M)^2$ gives the gauge invariant result (3)

$$C_\mu \simeq i(2\pi)^2 \frac{egG}{3M} [k_\mu(\mu^2 + pk) + p_\mu(m_K^2 + pk)] , \tag{3}$$

G is a constant related to the rate R_Λ (4) of the process $\Lambda^0 \to p + \pi$ through

$$G^2 = \frac{4\pi}{Q} \frac{m_\Lambda^2}{[(m_\Lambda^2 - M^2) - \mu^2(m_\Lambda^2 + m^2)]} R_\Lambda ,$$

(Q is the 3-momentum of the π^- in the Λ^0 rest system).

Finally one gets ($\alpha = e^2/4\pi$)

$$R_{e^+e^-} = \alpha^2 \left(\frac{g_{\Lambda K p}}{4\pi}\right)^2 \frac{1}{108\pi^3} \left(\frac{\mu}{M}\right)^4 \frac{m_K}{Q} F_1 F_2 ,$$

where

$$F_1 = \frac{m_\Lambda^2 M^2}{[(m_\Lambda^2 - M^2) + \mu^2(m_\Lambda^2 + M^2)]} ,$$

$$F_2 = \frac{1}{4}\left(\frac{m_K}{\mu}\right)^2 \left[\left(\frac{m_K^2 - \mu^2}{2m_K\mu}\right)^2 - \frac{3}{2}\right] + \frac{3}{2} \ln \frac{m_K}{\mu} .$$

The branching ratio is (5)

$$\frac{K^+ \to e^+ + e^- + \pi^+}{K^+ \to \text{all other modes}} = 1.0 \cdot 10^{-7} .$$

This ratio is indeed very small, smaller than it would be thought on the

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(2) N. CABIBBO and R. GATTO: *Phys. Rev.*, **116**, 1334 (1959); see also S. OKUBO. *Nuovo Cimento*, **16**, 963 (1960).

(3) The terms of higher order introduce an error $\sim 4\%$. The error induced by neglecting in the integrals the Λ-p mass difference is much smaller. M = nucleon mass, μ = pion mass.

(4) *High Energy Particle Data*, UCRL, vol. **2**, p. 85.

(5) The order of magnitude of $g_{\Lambda K p}$ can be inferred from dispersion relations: P. T. MATTHEWS and A. SALAM: *Phys. Rev.*, **110**, 769 (1958).

basis of an order of magnitude calculation (6). This is due to the « selection rule » provided by the mentioned theorem for the $(K^+\pi^+\gamma)$ vertex.

3. – The rate of the reaction

$$K_S^0 \to 2\gamma \tag{4}$$

can be deduced by associating the interaction responsible for the decay of the K_S^0 into two charged pions with the interaction of these pions with the electromagnetic field. The process will be described by a diagram of this type.

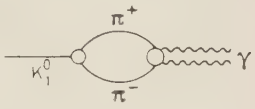

Fig. 3.

The left-hand side vertex will be associated to the rate of the normal $K_S^0 \to \pi^- + \pi^-$ decay; the right-hand side vertex can be safely evaluated by means of a standard perturbation calculation. The S-matrix element of the process is given by the following expression:

$$\langle k_1 k_2 | p \rangle = \frac{1}{\sqrt{2}} e^2 G \delta^4(p - k_1 - k_2)(2\pi)^{-9/2}(2EE_1E_2)^{-\frac{1}{2}}(\varepsilon_1\varepsilon_2)I\,,$$

G is a constant which is connected to the rate R_K of the $K_1^0 \to \pi^+ + \pi^-$ decay by

$$G^2 = 16\pi m_K^2 (m_K^2 - \mu^2)^{-\frac{1}{2}} R_K\,,$$

p, k_1,k_2 are the 4-momenta of the K_S^0 and of the γ's; E, E_1, E_2 are the corresponding energies. In the K_S^0 rest-system we have $E = M_K$, $E_1 = E_2 = \frac{1}{2}M_K$ ε_1, ε_2 are the polarization vectors of the γ's. The factor $1/\sqrt{2}$ comes from the symmetrization of the final state. The expression $(\varepsilon_1 \cdot \varepsilon_2)I$ is the result of the evaluation of the following integral:

$$(\varepsilon_1\varepsilon_2)I = \int d^4q \frac{4(\varepsilon_1 q)(\varepsilon_2 q)}{(q^2+\mu^2)[(q-k_1)^2+\mu^2][(q+k_2)^2+\mu^2]} +$$

$$+ \int d^4q \frac{4(\varepsilon_1 q)(\varepsilon_2 q)}{(q^2+\mu^2)[(q-k_2)^2+\mu^2][(q+k_1)^2+\mu^2]} - \int d^4q \frac{(\varepsilon_1\varepsilon_2)}{(q^2+\mu^2)[(q-k_1-k_2)^2+\mu^2]}\,.$$

Since the integrals are logarithmically divergent, a cut-off parameter must be introduced. Due to the fact that the mass of the intermediate state is less than the mass of the initial state, the integrals will not be pure imaginary

(6) An order of magnitude estimate was made by R. H. Dalitz: *Phys. Rev.*, **99**. 915 (1955): the branching ratio expected was $5 \cdot 10^{-5}\,(C/A)^2$, where C and A are two unknown constants. Comparison with the present result shows that $C/A = 1/20$.

(as in the usual evaluation of Feynman diagrams) but will have also a real part, which in the present case, however, turns out to be small.

The final expression of the rate $R_{2\gamma}$ for process (4) is ($\alpha = e^2/4\pi$)

$$R_{2\gamma} = \frac{1}{8} \frac{\alpha^2}{(2\pi)^3} G^2 \frac{|I|^2}{m_K} = \alpha^2 R_K N \,. \tag{5}$$

This result is of the expected order of magnitude, since the coefficient N is of the order unity with a cut-off of reasonable order of magnitude. For a cut-off equal to the nucleon mass M, we get $N = 0.37$: the variation of this number with the cut-off is very slow. The branching ratio of this process is

$$\frac{K_S^0 \to 2\gamma}{K_S^0 \to \text{all other modes}} \simeq 2.3 \cdot 10^{-5} \,.$$

As it has been said in the Introduction, the corresponding ratio for the K_L^0 is expected to be of the same order of magnitude. A calculation of the preceding type cannot be easily performed because of the presence of the unknown 3π - 2γ vertex.

* * *

We thank Prof. R. GATTO for helpful discussion.

RIASSUNTO

In vista di futuri esperimenti con fasci di K di alta intensità vengono valutati i rapporti di decadimento per $K^{\pm} \to \pi^{\pm} + e^+ + e^-$ e $K_1^0 \to 2\gamma$.

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

Effect of a Pion-Pion Scattering Resonance on Low Energy Meson-Photoproduction.

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(ricevuto il 5 Agosto 1960)

Summary. — The influence of the two and three pion intermediate states in the channel $\gamma\pi \to \mathcal{N}\mathcal{N}$ on photoproduction is studied. Application of the Cini-Fubini version of the Mandelstam representation to the amplitude $H^{(0)}$ (isoscalar photon) permits the evaluation of the contribution of a $J=I=1$ pion-pion resonance in terms of the isovector nucleon form factors and of the quantities describing the $\gamma+\pi \to \pi+\pi$ amplitude. In the limit of a very narrow resonance the result can be obtained by introducing an isovector vector boson coupled to $\gamma\pi$ and to $\mathcal{N}\overline{\mathcal{N}}$. This contribution can be merely added to the previous evaluations of $H^{(0)}$, since rescattering corrections are negligible at low energies. Arguments are given to show that the only important three-pion state is a resonant $I=0$, $J=1$ state which is crudely described by an isoscalar vector boson. Some of the parameters entering in this calculation can be obtained from experimental information on the isoscalar nucleon form factors. Though the three pion contribution to $H^{(+)}$ (isovector photon) is very simple, the final result is obtained in a more complicated way because rescattering corrections are important. Through crossing, $H^{(-)}$ (isovector photon) is also affected. It seems that one of the most interesting quantities to consider, in order to compare theory and experiment, is the $\mathrm{d}\sigma^+/\mathrm{d}\sigma^-$ ratio and its dependence on energies and angles.

1. – Introduction.

A good deal of interest has recently been devoted to the study of the pion pion interaction [1]. Although no direct evidence for such an interaction has as yet been found, indications of its existence have appeared, for example,

[1] G. Chew and S. Mandelstam: to be published in *Phys. Rev.*

in the analysis of single pion production in π-p collisions ([1-3]) and of $\mathcal{N}\bar{\mathcal{N}}$ annihilation ([4-5]).

More specifically, recent work by FRAZER and FULCO ([6]) has indicated that the experimental data on the isovector part of the nucleon electromagnetic form factors may be satisfactorily fitted by assuming a pion-pion scattering resonance in the $I = 1$, $J - 1$ state. In order to obtain additional confirmation of this hypothesis as well as further information regarding the values of the parameters characterizing the $J = I = 1$ resonance, it is of interest to consider the effect of this resonance on various observable processes. This has already been done for low energy pion-nucleon scattering ([7]) and is currently being done for nucleon-nucleon scattering ([8]). The present paper is concerned with the effect of this pion-pion resonance on pion photoproduction on nucleons (*). It is hoped, furthermore, that the existing discrepancies between the experimental data for this process and the theoretical predictions of CHEW, LOW GOLDBERGER and NAMBU ([9]) could be adequately explained by taking the pion pion resonance into account.

In Section **2** we recall the kinematics and write down the fundamental invariant forms as given by CGLN. Section **3** is concerned with the two and three pion intermediate states in the $\gamma\pi \to \mathcal{N}\bar{\mathcal{N}}$ reaction channel. In Section **4** the unitarity condition in this channel is applied to the isoscalar amplitudes and leads to expressions for their imaginary parts in terms of $\gamma\pi \to \pi\pi$ amplitude and the imaginary parts of the isovector nucleon form factors. In Section **5** the expressions for the isoscalar amplitudes resulting from the Cini-Fubini form of the Mandelstam representation combined with unitarity in channel III are written down under the assumption that the $I = \frac{1}{2}$ pion-nucleon phase shifts may be neglected.

In Section **6** it is shown that a simple model involving a $J = I = 1$ intermediate particle or « bipion » can reproduce the main features of the disper-

(*) Similar work has been carried out independently by MUNCZEK at the University of Roma. After this work was completed, we received a copy of a thesis by J. S. BALL (Lawrence Radiation Laboratory) who has obtained results similar to ours. *Phys. Rev. Lett.*: **5** 73 (1960).

([2]) F. BONSIGNORI and F. SELLERI: *Nuovo Cimento*, **15**, 465 (1960).

([3]) I. DERADO: *Nuovo Cimento*, **15**, 253 (1960).

([4]) G. GOLDHABER, W. FOWLER, S. GOLDHABER, T. HOANG, TH. KALOGEROPOULOUS, and W. POWELL: *Phys. Rev. Lett.*, **3**, 181 (1959).

([5]) F. CERULUS: *Nuovo Cimento*, **10**, 827 (1959).

([6]) W. FRAZER and J. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959); **117**, 1609 (1960).

([7]) J. BOWCOCK, N. COTTINGHAM and D. LURIÉ: *Nuovo Cimento*, **16**, 918 (1960); A. EFRENOV, V. MESHCHERYAKOV and D. SHIRKOV: preprint Dubna D-503.

([8]) D. AMATI, E. LEADER and B. VITALE: to be published in *Nuovo Cimento*.

([9]) G. CHEW, M. GOLDBERGER, G. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1345 (1957), designated hereunder by CGLN.

sion treatment of the preceding section. An analogous model, involving a $J = 1$, $I = 0$ intermediate particle or « tripion » is applied in Section **7** to the isovector amplitudes. In the hope of obtaining a rough estimate of some of the parameters characterizing this particle, the same model is applied to the isoscalar nucleon form factors. Finally a programme for the comparison of this theory with experiment is indicated in Section **8**.

2. – Decomposition in scalar amplitudes.

In this section we recall the spin and isotopic spin structure of the S-matrix element for photomeson production. We use the same notations as CGLN.

We first define three invariant scalar quantities:

$$\begin{aligned} s_1 &= -(k + p_1)^2 , \\ s_2 &= -(k - p_2)^2 , \\ t &= -(p_1 - p_2)^2 , \end{aligned}$$

where on the mass shell

$$s_1 + s_2 + t = 2M^2 + \mu^2 .$$

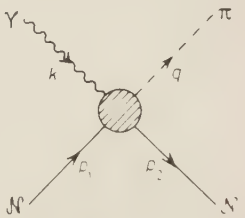

Fig. 1.

The various four momenta are defined in Fig. 1.

The S-matrix element can be written as

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta_4(k + p_1 - q - p_2) \frac{1}{(2\pi)^6} \frac{M}{2(k^0 q^0 p_1^0 p_2^0)^{\frac{1}{2}}} T_{fi} , \tag{1}$$

where the fields are normalized in the volume $(2\pi)^3$.

Because of the non-conservation of isotopic spin, one must introduce three invariant quantities for the photoproduction of one π-meson with isospin α:

$$\mathcal{T}_\alpha^+ = \delta_{3\alpha} , \qquad \mathcal{T}_\alpha^- = \tfrac{1}{2}[\tau_\alpha, \tau_3] , \qquad \mathcal{T}_\alpha^{(0)} = \tau_\alpha . \tag{2}$$

The first two correspond to the isovector part of the electromagnetic current and the latter to the isoscalar part.

As shown by CGLN, the requirements of Lorentz and gauge invariance lead to four fundamental forms which may be chosen to be

$$\begin{cases} M_A = i\gamma_5(\gamma e)(\gamma k) , \\ M_B = 2i\gamma_5[(e \cdot P)(q \cdot k) - (e \cdot q)(P \cdot k)] , \\ M_C = \gamma_5[(\gamma \cdot e)(q \cdot k) - (\gamma \cdot k)(q \cdot e)] , \\ M_D = 2\gamma_5[(\gamma \cdot e)(P \cdot k) - (\gamma \cdot k)(e \cdot P) - iM(\gamma \cdot e)(\gamma \cdot k)] , \end{cases} \tag{3}$$

where e is the photon polarization and P the combination

$$P = \frac{p_1 + p_2}{2} .$$

Writing now T_{fi} as

$$T_{fi} = \overline{u}_f(p_2)[\mathcal{T}_\alpha^{(+)} H^+ + \mathcal{T}_\alpha^{(-)} H^- + \mathcal{T}_\alpha^{(0)} H^0] u_i(p_1) , \tag{4}$$

where $u(p)$ is a Dirac spinor normalized according to $\overline{u}u = 1$, each function H may be decomposed as follows:

$$H = A(s_1 s_2 t) M_A + B(s_1 s_2 t) M_B + C(s_1 s_2 t) M_C + D(s_1 s_2 t) M_D . \tag{5}$$

Invariance of the T-matrix under the exchange of the two nucleons yields the following crossing properties: the scalar functions $A^{(-,0)}$, $B^{(+,0)}$, $C^{(-)}$, $D^{(+,0)}$ are even under the transformation $s_1 \rightleftarrows s_2$, $t \rightleftarrows t$ whereas $A^{(-)}$, $B^{(-)}$, $C^{(+,0)}$, $D^{(-)}$ are odd.

3. – Intermediate states for the reaction $\gamma + \pi \to \mathcal{N} + \overline{\mathcal{N}}$.

The combination of the Mandelstam representation with the unitarity condition for the reaction $\gamma + \pi \to \mathcal{N}\mathcal{N}'$ will enable us to take into account the effect of a resonant π-π interaction on the invariant photo-meson production amplitudes. Let us consider specifically the two- and three-pion intermediate states (*) which occur in the unitarity condition.

3·1. *Two-pion state.* – In a previous paper ([10]), it was shown that for the process $\gamma + \pi \to \pi + \pi$ only the isoscalar part of the photon contributes to the reaction amplitude. It follows that only the isoscalar amplitudes $A^{(0)}$, $B^{(0)}$, $C^{(0)}$ and $D^{(0)}$ will be affected. We note that the next highest state which contributes to these amplitudes is a four pion state (owing to G-invariance).

The two pions are in a $I = 1$ isospin state and their total angular momentum J is odd. As a result the existence of a $\pi\pi$ interaction in the S-state ([11]) will not affect the amplitudes under consideration. We will assume a $I = J = 1$ $\pi\pi$ resonance and neglect all other angular momentum states.

(*) The one-pion intermediate state has already been included in the Born terms given in CGLN in order to ensure gauge invariance.

([10]) M. GOURDIN and A. MARTIN: *Nuovo Cimento*, **16**, 78 (1960); See also H. S. WOUG: *Phys. Rev. Lett.*, **5**, 70 (1960).

([11]) N. KHURI and S. TREIMAN: preprint; R. SAWYER and K. WALI: preprint.

3·2. *Three-pion state.* – It can be seen from G invariance that only the isovector part of the photon current contributes to the $\gamma+\pi\to\pi+\pi+\pi$ reaction amplitude. Here the three pions have either $I=0$ or $I=1$. In the case where $I=0$ only the amplitudes associated with $\mathscr{T}_\alpha^{(+)}$ are involved whereas for $I=1$ only those associated with $\mathscr{T}_\alpha^{(-)}$ are affected.

It has been shown by CHEW and MOTTELSON [12] that in an $I=0$, $J=1$ three-pion state all three pairs of pions may be in an $I=J=1$ state. Conversely, if one requires that all three pairs of pions be in an $I=J=1$ state it can be shown that their total isospin and total angular momentum will be $I=0$ and $J=1$. The assumption made above of a strong pion-pion resonance leads us to retain only the three pion state characterized by these quantum numbers. As a result, the three pion state will affect only the amplitudes $A^{(+)}$, $B^{(-)}$, $C^{(+)}$ and $D^{(+)}$.

4. – Unitarity for the isoscalar amplitudes.

In the photoproduction channel, it is well known that the phases of the multipole amplitudes are given by the pion-nucleon scattering phase shifts in the corresponding quantum states. As the isoscalar amplitudes lead to a final state where $I=\frac{1}{2}$, and as the corresponding phase shifts are small, one may, at low energy, neglect the imaginary part of $A^{(-)}$, $B^{(0)}$, $C^{(0)}$ and $D^{(0)}$ following CGLN. At higher energies, of course, (above the $\frac{3}{2}\frac{3}{2}$ resonance, for example) such an approximation is no longer valid.

We now consider the reaction $\gamma+\pi\to\mathscr{N}+\mathscr{N}$. Introducing the centre of mass variables

$$k=(\boldsymbol{K},K)\,,\qquad q=(\boldsymbol{K},-\omega_K)\,,\qquad p_1=(-\boldsymbol{p},-E)\,,\qquad p_2=(\boldsymbol{p},-E)\,,$$

and setting $e_4=0$, $e\cdot K=0$ by a suitable choice of gauge, we can write the isoscalar amplitude as a matrix in the nucleon spin space as follows:

$$T=i\frac{K}{M}(A+tB)(\boldsymbol{e}\cdot\boldsymbol{p})+\frac{2KE}{M}C(\boldsymbol{\sigma},\boldsymbol{e},\boldsymbol{p})+\left(A-\frac{t}{2M}D\right)(\boldsymbol{\sigma},\boldsymbol{e},\boldsymbol{K})+\\+\frac{1}{M(M+E)}(A+2ED)(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{e},\boldsymbol{K},\boldsymbol{p})\,, \tag{6}$$

where the index $^{(0)}$ has been dropped; A, B, C and D are functions of $t=4E^2$ and $\cos\varphi=\boldsymbol{K}\cdot\boldsymbol{p}/Kp$.

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[12] G. CHEW: *Phys. Rev. Lett.*, **4**, 142 (1960).

The unitarity condition in this channel may be written as

$$\langle f|T-T^{+}|i\rangle=\sum_{n}\langle f|T|n\rangle\langle n|T^{+}|i\rangle\,. \tag{7}$$

In the energy region $(2\mu)^2\leqslant t\leqslant(2M)^2$ equation (7) is to be understood as an analytic continuation of the unitarity condition from the region of physical energies ([13]). Retaining only the two-pion contributions to the RHS of equation (7), one can express the imaginary parts (*) of A, B, C and D in terms of the $\gamma+\pi\to\pi+\pi$ and $\pi+\pi\to\mathcal{N}+\overline{\mathcal{N}}$ reaction amplitudes given in references ([10,7]).

We note, however, that the same $I=1$, $J=1$ two pion intermediate state occurs in the expression of the imaginary part of the isovector nucleon form factors ([14]). As the same $\pi+\pi\to\mathcal{N}+\overline{\mathcal{N}}$ amplitude appears therefore in both processes, we can eliminate this amplitude and re-express the imaginary parts of A, B, C and D in terms of the imaginary parts of the isovector nucleon form factors. More precisely the $\gamma\to\mathcal{N}+\overline{\mathcal{N}}$ vertex function may be written in the centre of mass system of the $\mathcal{N}\overline{\mathcal{N}}$ pair

$$\frac{E}{M}\,e_\mu\langle\mathcal{N}\overline{\mathcal{N}}|j_\mu|0\rangle=\chi^{*}_{\mathcal{N}}\Big\{\frac{E}{M}\,(\boldsymbol{\sigma}\cdot\boldsymbol{e})[F_1^v(t)+2M\,F_2^v(t)]+ \\ +\frac{1}{M(E+M)}\,(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{e}\cdot\boldsymbol{p})[2EF_2^v(t)-F_1^v(t)]\Big\}\chi_{\overline{\mathcal{N}}}\,, \tag{8}$$

where we have used the same notations as for the $\gamma+\pi\to\mathcal{N}+\mathcal{N}$ reaction. The invariant functions $F_1^v(t)$ and $F_2^v(t)$ are the ordinary isovector form factors for the nucleon. On the other hand, the $\gamma\to\pi_l+\pi_m$ vertex and the $\gamma+\pi_k\to\pi_l+\pi_m$ reaction amplitude are given respectively by ([6,14,10])

$$2\omega_q e_\mu\langle\pi_l\pi_m|j_\mu|0\rangle=e\varepsilon_{3lm}\boldsymbol{e}\cdot\boldsymbol{q}\,F_\pi(t)\,, \tag{9}$$

where the pion form factor $F_\pi(t)$ satisfies $F_\pi(0)=1$, and

$$2\omega\,(2\omega_K)^{\frac{1}{2}}\langle\pi_l\pi_m|j_\mu|\pi_n\rangle\,e_\mu=\frac{e\Lambda}{\mu^3}\,\frac{\varepsilon_{lmn}}{\sqrt{2}}\,\omega_s(\boldsymbol{e},\boldsymbol{K},\boldsymbol{q})\,F_\pi(t)\varphi(t)\,, \tag{10}$$

where $\varphi(t)$ is a real smooth functions considered in reference ([10]) and Λ a dimensionless « coupling constant » corresponding to the normalization $\varphi(0)=1$,

([13]) S. MANDELSTAM: *Phys. Rev. Lett.*, **4**, 84 (1960).

([14]) P. FEDERBUSH, M. GOLDBERGER and S. TREIMAN: *Phys. Rev.*, **112**, 642 (1958).

(*) The phase of M_A, M_B, M_C and M_D have been chosen so that the imaginary parts of A, B, C and D will appear in the left hand side of eq. (7).

The formal substitution

$$\boldsymbol{e} \to \boldsymbol{e} \times \boldsymbol{K} ,$$

allows one to establish a correspondence between equations (9) and (10) and equations (6) and (8) and to write

$$\text{(11)} \qquad \begin{cases} \operatorname{Im} A = \dfrac{1}{4\sqrt{2}} \dfrac{A}{\mu^3} t\varphi(t) \operatorname{Im} F_2^v(t) , \\ \operatorname{Im} B = -\dfrac{1}{4\sqrt{2}} \dfrac{A}{\mu^3} \varphi(t) \operatorname{Im} F_2^v(t) , \\ \operatorname{Im} C = 0 , \\ \operatorname{Im} D = -\dfrac{1}{4\sqrt{2}} \dfrac{A}{\mu^3} \varphi(t) \operatorname{Im} F_1^v(t) . \end{cases}$$

5. – Mandelstam representation for the isoscalar amplitudes.

In this section we again consider only the amplitudes $A^{(0)}$, $B^{(0)}$, $C^{(0)}$ and $D^{(0)}$ for which we assume Mandelstam representations

$$\text{(12)} \qquad \text{Born terms} + \frac{1}{\pi^2} \int\limits_{(M+\mu)^2}^{\infty} \int\limits_{(2\mu)^2}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \frac{\varrho(x, y)}{y - t} \left[\frac{1}{x - s_1} + \varepsilon \frac{1}{x - s_2} \right] + \\ + \frac{1}{\pi^2} \int\limits_{(M+\mu)^2}^{\infty} \int\limits_{(M+\mu)^2}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \frac{\sigma(x, y)}{(x - s_1)(y - s_2)} ,$$

where the Born terms may be deduced from those given in CGLN and where, as a result of crossing $\varepsilon\sigma(xy) = \sigma(yx)$ and $\varepsilon = 1$ for A, B and D and $\varepsilon = -1$ for C. It is easily seen that owing to the isoscalar nature of the photon current, one may apply the Cini-Fubini arguments [15] and write the representation (12) in one dimensional form, where the corresponding weight functions are directly related to the absorptive parts of the amplitudes in the various channels. As mentioned in the previous section, one may neglect the absorptive parts in the photoproduction channels. As shown by equations (11) furthermore, the absorptive parts of the amplitudes in the $\gamma\pi \to \mathcal{N}\overline{\mathcal{N}}$ channel are simply functions of t in the approximation where only the $I = J = 1$ two-pion intermediate state is retained. These considerations lead us to write the

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[15] M Cini and S. Fubini: to be published in *Ann. Phys.*

H^0 amplitude (5) using (11) as follows:

$$H^0 = H^0_{\text{Born}} + \frac{1}{4\sqrt{2}}\frac{\Lambda}{\mu^3}\Big[M_A \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{t'\,\varphi(t')\,\operatorname{Im} F_2^v(t')\,\mathrm{d}t'}{t'-t} - \tag{13}$$

$$- M_B \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varphi(t')\,\operatorname{Im} F_2^v(t')\,\mathrm{d}t'}{t'-t} - M_D \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varphi(t')\,\operatorname{Im} F_1^v(t')\,\mathrm{d}t'}{t'-t}\Big],$$

where possible subtractions have been omitted. The form of the first integral in equation (13) would seem to suggest that a subtraction might be effected so as to cast this integral into the same form as the second one. If one does this, equation (13) becomes

$$H^0 = H^0_{\text{Born}} + \frac{1}{4\sqrt{2}}\frac{\Lambda}{\mu^3}\Big[A_0 M_A + (tM_A - M_B)\frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varphi(t')\,\operatorname{Im} F_2^v(t')\,\mathrm{d}t'}{t'-t} - \tag{14}$$

$$- M_D \frac{1}{\pi}\int_{(2\mu)^2}^{\infty} \frac{\varphi(t')\,\operatorname{Im} F_1^v(t')\,\mathrm{d}t'}{t'-t}\Big].$$

At this point we make the assumption of a sharp pion-pion resonance. Writing the $J=1$, $I=1$ pion-pion scattering amplitude as

$$\exp[i\delta]\sin\delta = \frac{\gamma q^3}{t_R - t - i\gamma q^3}, \tag{15}$$

where t_R is the square of the total c.m. resonance energy of the two pions and where γ is proportional to the resonance width, we shall make the approximation of replacing $\sin^2\delta$ by $\pi\gamma q_R^3\,\delta(t_R - t)$. With this approximation it has been shown in reference (7) that the imaginary parts of the isovector nucleon form factors are of the form

$$\begin{cases} \operatorname{Im} F_1^v(t) = -\dfrac{e}{2}\pi a t_R\,\delta(t - t_R), \\ \operatorname{Im} F_2^v(t) = -\dfrac{eg}{2M}\pi b t_R\,\delta(t - t_R), \end{cases} \tag{16}$$

where g is the isovector part of the anomalous gyromagnetic ratio. The constants a and b are related to γ, t_R, and two new constants C_1 and C_2 describing in the same approximation the $J=1$, $I=1$ $\pi\pi \to \mathcal{N}\bar{\mathcal{N}}$ reaction

amplitude:

$$a = \frac{2C_1}{\gamma}\frac{t_R-\gamma}{(t_R)^{\frac{3}{2}}}, \qquad b = \frac{M}{g}\frac{2C_2}{\gamma}\frac{t_R-\gamma}{(t_R)^{\frac{3}{2}}}. \tag{17}$$

Insertion of (16) into (14) yields the simple form

$$H^0 = H^0_{\text{Born}} + \frac{1}{8\sqrt{2}}\frac{e\Lambda}{\mu^3}\, t_R\varphi(t_R)\left[A_0' M_A - \frac{gb}{M}\frac{tM_A - M_B}{t_R - t} + a\frac{M_D}{t_R - t}\right]. \tag{18}$$

We have thus obtained an expression for the isoscalar amplitude in terms of the following parameters:

— a, b and t_R which are linked to the nucleon form factors or, equivalently, to the $\pi+\pi \to \mathcal{N}+\bar{\mathcal{N}}$ reaction amplitudes ([7]). A possible set of values for these three parameters has been given in reference ([7]) by comparison of the experimental data with the nucleon form factors.

— Λ which occurs in the expression for the $\gamma+\pi \to \pi+\pi$ reaction amplitude ([10]). One might hope to estimate this constant by comparison with experimental data for photomeson production on nucleons or eventually Compton scattering on nucleons. Alternatively, this constant could be measured directly, at least in principle, by the procedures discussed in reference ([10]).

— An adjustable parameter A_0'. The use of further subtractions would, of course, introduce additional parameters of this type.

The simplicity of the present treatment is essentially due to our neglect of the absorptive parts of the isoscalar multipole amplitudes in channel I. Had we taken these absorptive parts into account we would, of course, have been led to a system of coupled Omnès-type integral equations in the multipole amplitudes for H^0.

6. – « Bipion » model.

It is of interest to note that the results of the preceding section in the case of a narrow resonance may be obtained quite simply by means of an elementary first order perturbation calculation. We can indeed introduce a new intermediate particle B which we will take to simulate the effect of a narrow $J=1$, $I=1$ pion-pion resonance in the intermediate state. This particle whose mass will be taken to be $\sqrt{t_R}$ must, of course, be a vector boson as well as a vector in isotopic spin space.

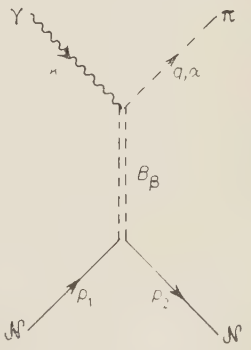

Fig. 2.

More specifically, we shall compute the graph drawn in Fig. 2.

The $\gamma\pi$B vertex may be written as

$$\lambda\delta_{\alpha\beta}\varepsilon_{\mu\nu\varrho\sigma}e_{\mu}k_{\nu}q_{\varrho} \tag{19}$$

by Lorentz and gauge invariance.

The $\mathcal{N}\overline{\mathcal{N}}$B vertex can be shown to contain two types of couplings:

$$\overline{u}(p_2)\left[\mathscr{C}_1\gamma_\tau+\frac{i}{2}\mathscr{C}_2[\gamma_\tau,\gamma_\lambda](p_2-p_1)_\lambda\right]\tau_\beta u(p_1)\,. \tag{20}$$

A third coupling proportional to the divergence of the B field is *a priori* possible but may be shown to give no contribution.

The boson propagator ([16]) is proportional to

$$\frac{\delta_{\varrho\tau}-[(q-k)_\varrho(q-k)_\tau/t_R]}{t_R-t} \tag{21}$$

(we recall that $t=-(q-k)^2$); by combination of (19) with (21) one immediately sees that the second term in the boson propagator does not contribute to the matrix element under consideration. After some algebraic manipulations the final result can be expressed in terms of the fundamental forms appearing in (3):

$$H^0_{\text{bipion}}=\lambda\left[\frac{\mathscr{C}_2}{t-t_R}(tM_A-M_B)-\frac{\mathscr{C}_1}{t-t_R}M_D\right]. \tag{22}$$

Inspection of equations (17), (18) and (22) shows that they are of the same form if one takes $A'_0=0$ and

$$\frac{\mathscr{C}_1}{\mathscr{C}_2}=\frac{C_1}{C_2}\,. \tag{23}$$

It is of some interest to note that by using the « bipion » model we are able to reproduce the main features of the dispersion treatment given in the previous section under the assumption of a narrow pion pion resonance.

7. – Isovector amplitudes: tripion model.

If one now considers the Mandelstam representation for the isovector amplitudes $A=B^{\pm}C^{\cdot}D^{+}$ one finds that owing to G invariance one cannot directly carry through the Cini-Fubini reduction. Indeed, one sees that for the graph

([16]) R. P. FEYNMAN: *Phys. Rev.*, **76**, 769 (1949).

drawn in Fig. 3-*a* the variables s_1 and t may both reach their lower limits of integration in the Mandelstam integrals. The same situation holds for the variables s_1 and s_2 if one considers the graph drawn in Fig. 3-*b*. On the other hand, nothing is as yet known regarding the $\gamma+\pi\to\pi+\pi+\pi$ and $\pi+\pi+\pi\to\overline{\mathcal{N}}+\mathcal{N}$ production amplitudes.

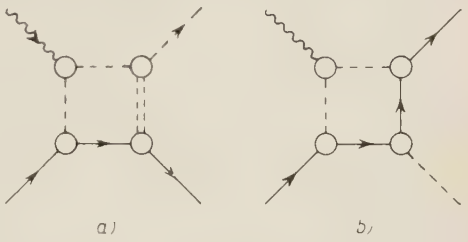

Fig. 3.

Under these circumstances, we shall consider a very naive model in which the three pion intermediate state is replaced by the tripion particle considered in Section **3**, with quantum numbers $J=1$, $I=0$. As we are again dealing with a vector boson, we may write, in complete analogy with Section **6**,

$$H^{+}_{\text{tripion}}=\lambda'\left[\frac{\mathscr{C}'_2}{t-t'_R}\,(tM_A-M_B)-\frac{\mathscr{C}'_1}{t-t'_R}\,M_D\right], \tag{24}$$

where λ' is the coupling constant corresponding to the $\gamma\pi$ tripion vertex and where $\mathscr{C}'_1$ and $\mathscr{C}'_2$ are the coupling constants of the tripion to the nucleon; t'_R is the square of the tripion mass.

In order to get some information about the parameters appearing in formula (24), we will apply a similar model to the calculation of the isoscalar nucleon form factors, and assume that these may be written as a sum of two terms, the first coming from the graph $\gamma\to$ tripion $\to\mathcal{N}+\mathcal{N}$, and the second from graphs comprising higher mass intermediate states; we will consider the latter contributions to be adequately represented by additive constants:

$$\begin{cases} F_1^s(t)=\dfrac{e}{2}\left[1-\dfrac{a't}{t'_R-t}\right], \\ F_2^s(t)=\dfrac{g'e}{2m}\left[1-\dfrac{b't}{t'_R-t_R}\right], \end{cases} \tag{25}$$

where g' is the isoscalar part of the anomalous gyromagnetic ratio, and where

$$\frac{b'}{a'}=\frac{m}{g'}\frac{\mathscr{C}'_2}{\mathscr{C}'_1}. \tag{26}$$

Experimentally $F_1^s(t)$ and $F_1^v(t)$ are rather close in value owing to the smallness of the neutron charge form factor. We shall therefore take

$$a'\simeq a\,;\qquad t'_R\simeq t_R\,, \tag{27}$$

where the strict equality would correspond to $F_1^N(t)\quad 0$. On the other hand,

for values of t between $-25\mu^2$ and 0 we have [17]

$$0.7 < \frac{F_2^N(t)/F_2^N(0)}{F_2^P(t)/F_2^P(0)} < 1$$

These experimental values lead to the following limits on the value of the ratio $\mathscr{C}_2'/\mathscr{C}_1'$:

$$0.5 < \frac{m}{g'}\frac{\mathscr{C}_2'}{\mathscr{C}_1'} < 5 \,. \tag{28}$$

Returning now to the photoproduction amplitude, it is clear that one cannot simply add on the tripion contribution to the solution obtained by CGLN for H^+ owing to the presence of the $J=\frac{3}{2}$, $I=\frac{3}{2}$ pion-nucleon final state interactions in the isovector amplitudes.

If we now make the tripion approximation, the cut in the t plane of the amplitudes is replaced by a pole at $t=t_K$, so that only the cuts in s_1 and s_2 remain. The amplitude H^+ may then be written

$$H^+ = H^+_{\text{Born}} + H^+_{\text{tripion}} + \sum_{ABCD} M_A \frac{1}{\pi^2} \int_{(m+\mu)^2}^{\infty} \int_{(m+\mu)^2}^{\infty} \frac{\varrho_A(s_1' s_2')\, \mathrm{d}s_1' \mathrm{d}s_2'}{(s_1' - s_1)(s_2' - s_2)} \,. \tag{29}$$

Taking a fixed value of t we can reduce the double integral to a sum of two one-dimensional integrals and write for A^+, for example:

$$A^+(s_1 t) = A^+_{\text{Born}}(s_1 t)\; A^+_{\text{tripion}}(t) + \\ + \frac{1}{\pi} \int_{(m+\mu)^2}^{\infty} \mathrm{d}s'\, \mathrm{Im}\, A^+(s' t) \left\{ \frac{1}{s' - s_1 - i\varepsilon} + \frac{1}{s' - 2m^2 - \mu^2 + t + s_1} \right\} . \tag{30}$$

Similar representations satisfying crossing symmetry hold for B^+, C^+ and D^+. Note that as a result of inequality (28) and of equation (24), the tripion contribution will mainly affect the D^+ amplitude.

It may easily be seen that the amplitudes $A^- B^- C^- D^-$ satisfy representations of the same form with the exception that the tripion terms will be absent; these are simply the equations written down by CGLN.

When unitarity is taken into account, this representation leads to a system of coupled Omnès type integral equations in the multipoles, involving as a

[17] *Proc. of the Annual International Conference on High Energy Physics* (CERN, 1958), edited by B. FERRETTI.

result of crossing both the (—) and (+) amplitudes. By well known techniques ([18]), this system may be transformed into a system of Fredholm-type integral equations. These considerations will be deferred to a succeeding paper.

8. – Programme for a comparison with experiment and conclusions.

The amplitudes for photoproduction of mesons with definite charge are linear combinations of T^+, T^- and T^0:

$$
(31)\qquad \begin{cases} \gamma + \mathrm{p} \to \mathrm{n} + \pi^+ & \sqrt{2}(T^- + T^0)\,, \\ \gamma + \mathrm{n} \to \mathrm{p} + \pi^- & -\sqrt{2}(T^- - T^0)\,, \\ \gamma + \mathrm{p} \to \mathrm{p} + \pi^0 & T^+ + T^0\,, \\ \gamma + \mathrm{n} \to \mathrm{n} + \pi^0 & T^- - T^0\,. \end{cases}
$$

We see that the amplitudes for production of charged mesons are mainly affected by the bipion term and the CGLN terms as the tripion contribution to T^- coming from crossing will be less important. The photoproduction of π_0's on the other hand will be affected by both the bipion and the tripion as well as by the CGLN terms.

The energy dependence of the experimentally measured ratio $\mathrm{d}\sigma^-/\mathrm{d}\sigma^+$ at 90° is in disagreement with the theoretical prediction based on the CGLN calculation. Whereas the theoretical curve is relatively flat, the experimental curve decreases steadily with energy in the 170 to 250 MeV region ([17]). Inspection of equation (31) shows that this ratio is extremely sensitive to the pion-pion resonance correction to T^0. It is hoped, therefore, that the comparison of our theory with the experimental data for this ratio measured at different angles may provide a severe test of the existence of a $J = 1$, $I = 1$ pion-pion resonance (*). It should be pointed out that since the experimental values of $\mathrm{d}\sigma^-/\mathrm{d}\sigma^+$ at 90° are of the order of 1 for the energies under consideration, and since T^- is affected by the $\frac{3}{2}\,\frac{3}{2}$ resonance, whereas T^0 is not, the ratio $|T^- - T^0|^2/|T^- + T^0|^2$ will be relatively insensitive to the small tripion correction to T^-. As this latter correction can only be evaluated by means of a very simplified model, this circumstance is rather fortunate insofar as the comparison of this aspect of our theory with experiment is concerned.

([18]) R. Omnès: *Nuovo Cimento*, **8**, 316 (1958).

(*) According to Munczek $\mathrm{d}\sigma^-/\mathrm{d}\sigma^+$ is indeed improved by the inclusion of the bipion term but there are still some difficulties with the π^+ photoproduction cross-section itself.

For the comparison of our theory with the experimental results on π^0 production, one could in principle hope to fit the existing experimental curves for the differential cross-sections by means of expressions of the type

$$a + b\cos\theta + c\cos^2\theta + \frac{d}{t - t_R} + \frac{e}{(t - t_R)^2},$$

assuming $t_R = t'_R$, and where e is expressed in terms of the bipion and tripion parameters. An analogous method was used by MORAVCZIK to isolate the retardation term in the case of charged meson photoproduction ([19]). In our case, however, this method does not seem too hopeful at first sight owing to the large distance of the pole $t = t_R$ from the physical region at low energies.

Another method of comparison would involve solving the integral equations for T^+ and T^- and comparing directly with the experimental total and differential cross-sections.

* * *

It is a pleasure to thank Professor S. FUBINI for his continued interest in this work and Dr. MUNCZEK for a fruitful discussion.

([19]) J. TAYLOR, M. MORAVCSIK and J. URETSKY: *Phys. Rev.*, **113**, 689 (1959).

RIASSUNTO (*)

Si studia l'influenza dello stato intermedio a due e tre pioni nel canale $\gamma\pi \to \mathcal{N}\bar{\mathcal{N}}$ sulla fotoproduzione. L'applicazione della versione Cini-Fubini della rappresentazione di Mandelstam all'ampiezza $H^{(0)}$ (fotone isoscalare) permette di valutare il contributo di una risonanza $J = I = 1$ pione-pione in termini dei fattori di forma isovettoriali del nucleone e delle quantità che descrivono l'ampiezza $\gamma + \pi \to \pi + \pi$. Nel limite di una risonanza molto stretta il risultato può essere ottenuto introducendo un vettore bosonico isovettoriale accoppiato a $\gamma\pi$ ed a $\mathcal{N}\bar{\mathcal{N}}$. Questo contributo può essere semplicemente aggiunto alla precedente valutazione di $H^{(0)}$, poichè le correzioni di riscattering sono trascurabili a basse energie. Si danno prove per mostrare che il solo stato importante a tre pioni è uno stato risonante $I = 0$, $I = 1$ che è descritto grossolanamente da un vettore bosonico isovettoriale. Alcuni dei parametri introdotti in questi calcoli possono essere ottenuti da dati sperimentali sui fattori di forma nucleonici isoscalari. Sebbene il contributo tripionico a $H^{(+)}$ (fotone isovettoriale) sia molto semplice, il risultato finale si ottiene in modo più complicato perchè le correzioni di riscattering sono importanti. Tramite l'incrocio anche $H^{(-)}$ (fotone isovettore) viene influenzato. Sembra che una delle più interessanti quantità da considerare, per confrontare teoria ed esperimento, è il rapporto $d\sigma^+/d\sigma^-$ e la sua dipendenza dalle energie e dagli angoli.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

Bound States, Shadow States and Mandelstam Representation.

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(ricevuto il 5 Agosto 1960)

Summary. — In a previous paper a technique involving complex angular momenta was used in order to prove the Mandelstam representation for potential scattering. One of the results was that the number of subtractions in the transmitted momentum depends critically on the location of the poles (shadow states) of the scattering matrix as a function of the complex orbital momentum. In this paper the study of the position of the shadow states is carried out in much greater detail. We give also related inequalities concerning bound states and resonances. The physical interpretation of the shadow states is then discussed.

1. – Introduction.

The validity of the Mandelstam representation for the potential scattering of two spinless particles has now been firmly established for a class of generalized Yukawa potentials ([1-4]). Double dispersion relations can now be proved in a variety of methods each one adding an interesting angle to the overall picture of the analytic properties of the scattering amplitude.

In spite of these advances many details are missing and in particular little is known of the connection between bound states and subtraction terms in the transmitted momentum.

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([1]) T. Regge: *Nuovo Cimento*, **14**, 951 (1959).

([2]) R. Blankenbecler, M. L. Goldberger, M. N. Khuri and S. B. Treiman: *Ann. of Phys.*, **10**, 62 (1960).

([3]) A. Klein: *Journ. Math. Phys.*, **1**, 41 (1960).

([4]) L. Fonda, L. A. Radicati and T. Regge: *Mandelstam representation for the non-relativistic many channel problem* (in proof).

BLANKENBECLER *et al.* [2] have shown how unitarity and Mandelstam representation can replace the Schrödinger equation in the construction of the scattering matrix under some simplifying hypothesis on bound states and subtraction terms. Their analysis would attain full generality and beauty if these restrictions could be removed and their statements rigorously backed.

In this respect part of the gap is filled in a previous paper [1] of the author where it is shown that, under some restrictive hypothesis on the potential, the number of needed subtractions is finite.

In the present work explicit bounds are placed on L and some restrictions on the potential are removed. Similar bounds are also derived for bound states. Finally a simple physical interpretation is given for the shadow states.

2. – Definitions and previous results.

In [1] complex angular momenta were employed in the proof of Mandelstam representation. We summarize here briefly the most important definitions and results needed for the proof. The starting point is the partial wave equation

$$D(E, \beta)\psi = 0 \,, \tag{2.1}$$

where

$$D(E, \beta) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} + E + \left(\beta + \frac{1}{4}\right) x^{-2} - V(x) \,,$$

and

$$\beta = -(l + \tfrac{1}{2})^2 \,.$$

Eq. (2.1) will be considered for generally complex values of β. $l = \sqrt{-\beta} - \frac{1}{2}$ will be defined as that branch of the function which is one valued in the β plane with the cut $0 \leqslant \beta < \infty$ and such that $\mathrm{Re}\,(l+\frac{1}{2}) > 0$. The « regular » solution φ of eq. (2.1) is defined by the following boundary conditions:

$$\begin{cases} \varphi(E, \beta) \sim \cos\left[x\sqrt{E} - \dfrac{\pi\sqrt{-\beta}}{2} + \delta(\beta)\right], & x \to \infty \,, \\ \varphi(E, \beta) \sim C x^{l+1} \,, & x \to 0 \,, \end{cases} \tag{2.2}$$

φ defines therefore an asymptotic phase shift $\delta(\beta)$ which reduces to the physical values when l is integer. In [1] was shown that $S(\beta) = \exp[2i\delta(\beta)]$ has the following properties:

a) $S(\beta)$ is meromorphyic in the β plane with the exception of a discontinuity across the cut $0 \leqslant \beta < \infty$.

b) $S(\beta)S^+(\beta) = 1$ where $S^+(\beta) = (S(\beta^*))^*$. This is equivalent to $\delta(\beta) = \delta^+(\beta)$.

In particular $\delta(\beta)$ is real if β is negative.

c) If β is negative $\dfrac{\partial\delta(\beta)}{\partial\beta} > \dfrac{\pi}{2}\sqrt{-\beta}$.

d) $\operatorname{Im}\sqrt{-\beta}\,[\pi \operatorname{Im}\sqrt{-\beta} - 2 \operatorname{Im}\delta(\beta)] > 0$.

Condition *d*) implies that $S(\beta)$ can have poles in $\operatorname{Im}\beta < 0$ only. *c*) shows that if there is a pole in β there is a zero in β^*.

e) $\lim\limits_{|\beta|\to\infty} \exp[-i\pi\sqrt{-\beta}]\,[S(\beta) - 1] = 0$, $\qquad \arg\beta \geqslant 0$.

If in addition to (2.1) we suppose that $V(x)$ is a generalized Yukawa potential

$$V(x) = \int_{\mu}^{\infty} \mathrm{d}m\, \sigma(m)\, \frac{\exp[-mx]}{x}, \tag{2.3}$$

we find that *e*) can be improved as follows:

e')
$$\begin{cases} S(\beta) - 1 = 0 \left(\exp[i\varepsilon\sqrt{-\beta} - \alpha\sqrt{-\beta}]\right), & \arg\beta \geqslant 0, \\ \cosh\alpha = 1 + \dfrac{\mu^2}{2E}, & \varepsilon > 0 \text{ and small.} \end{cases}$$

Properties *a*) ... *e'*) are then used in connection with the partial wave expansion of the scattering amplitude

$$f(z) = \frac{1}{2i\sqrt{E}} \sum_l (2l+1)\left(\exp[2i\delta_l] - 1\right) P_l(z); \qquad z = \cos\theta. \tag{2.4}$$

This expansion is known to converge in the small Lehmann ellipse $\operatorname{Im}\theta < \alpha$. According to WATSON we can transform it into the integral

$$f(z) = \frac{1}{4\sqrt{E}} \int_C \mathrm{d}\beta [S(\beta) - 1] P_{\sqrt{-\beta} - \frac{1}{2}}(-z)\, \frac{1}{\cos\pi\sqrt{-\beta}}. \tag{2.5}$$

The contour C loops around the poles of $(\cos\pi\sqrt{-\beta})^{-1}$ which occur on the physical values of β on the negative axis. C must not enclose any singularity of $S(\beta)$. It is possible to deform C around the positive axis and transform

(2.5) into

$$\sqrt{E}f(z) = \frac{i\pi}{2}\sum_n S_n P_{l_n}(-z)(2l_n+1)\frac{1}{\sin \pi l_n} + \tag{2.6}$$

$$+\frac{1}{4}\int_0^\infty d\beta [S(\beta+i\varepsilon)-S(\beta-i\varepsilon)] P_{\sqrt{-\beta}-\frac{1}{2}} \frac{1}{\cos \pi\sqrt{-\beta}},$$

where S is the residuum of $S(\beta)$ in the pole $\beta_n = -(l_n+\frac{1}{2})^2$. The summation is carried out on all poles of $S(\beta)$. The functions $P_{l_n}(-\cos\theta)$, where l_n is not an integer, are Legendre functions. It is apparent from (2.6) and (2.4) that $f(\cos\theta)$ is analytic in the $\cos\theta$ plane with the cut $\cosh\alpha < \cos\theta < \infty$.

The contribution of the integral can be shown to vanish when $\operatorname{Im}\theta \to \infty$ (or $\cos\theta \to \infty$). The function $P_{l_n}(-\cos\theta)$ instead of the order of $\cos\theta^{\operatorname{Re} l_n}$. If the set is bounded above by L' the behavior of $f(\cos\theta)$ will be dominated by $\cos\theta^{L'}$. If $L' = \infty$ $f(\cos\theta)$ will have an essential singularity at $\cos\theta = \infty$. In (1) we proved that L' is finite if $|y^2 \operatorname{Im} V(iy)| < A < \infty$. In this paper we shall give an explicit estimate for L'. The number of subtractions needed in the Mandelstam representation is then the mallest integer $L > L'$. In (2) it is proved that unitarity requires $L > K$, where K is the largest angular momentum for which there is a bound state.

3. – Bound states.

A bound state is defined as a solution of the equation

$$D(E,\beta)\psi = 0 \qquad \text{where } E,\ \beta < 0;\ (l \text{ integer}) \tag{3.1}$$

which is L^2 in $0 \ldots \infty$. We shall name bound states the solutions of (3.1) with the same boundary conditions also when l is any positive number. L^2 implies the asymptotic behaviour

$$\begin{cases} \psi \sim c x^{l+1} & x \to 0\,, \\ \psi \sim \exp[-\sqrt{-E}x] & x \to \infty\,. \end{cases} \tag{3.2}$$

In order to have a bound state E and β must be related by some equation $g(E,\beta) = 0$, where g depends on V. This equation, if solved in E, yields $E = f(\beta)$ where f is a multivalued function since there may exist several bound states with the same β.

For definiteness we choose the branch with the largest binding energy.

It is clear that in general $f(\beta)$ is differentiable. Write now the equation

$$\int_0^\infty \mathrm{d}x\, \psi D(E,\beta)\psi = 0\,. \tag{3.3}$$

After an integration by parts and some algebra (3.3) becomes

$$\int_0^\infty \mathrm{d}x \left[\left(\psi' - \frac{\psi}{2x}\right)^2 + |E|\psi^2 + |\beta|\frac{\psi^2}{x^2}\right] = -\int_0^\infty V(x)\psi^2\,\mathrm{d}x\,. \tag{3.4}$$

A necessary condition for the existence of bound states is that

$$|E| + (|\beta|/x^2) + V(x)$$

is somewhere negative. This implies that somewhere

$$|E| + (|\beta|/x^2) \leqslant V(x).$$

This latter condition is of course the same as in classical mechanics if $l + \frac{1}{2}$ is the classical angular momentum.

If V is repulsive, *i.e.* $V \geqslant 0$, there are no bound states. Suppose now that V satisfies an inequality of the kind

$$|V(x)| \leqslant \frac{M_\alpha}{x^{2\alpha}}\,. \tag{3.5}$$

By derivation it can be checked that

$$P(\lambda) = \frac{|E|}{\lambda^\alpha} + \lambda^{\alpha-1}\frac{|\beta|}{x^2}\,, \tag{3.6}$$

satisfies the inequality

$$P(\lambda) \geqslant \alpha^{-\alpha}(1-\alpha)^{\alpha-1}|E|^{1-\alpha}\left|\frac{\beta}{x}\right|^\alpha,$$

putting $\lambda = 1$ one finds

$$|E| + \frac{|\beta|}{x^2} \geqslant \left|\frac{E}{1-\alpha}\right|^{1-\alpha}\left|\frac{\beta}{\alpha}\right|^\alpha x^{-2\alpha}\,.$$

Comparing this result with (3.4) we get

$$\left|\frac{\beta}{\alpha}\right|^\alpha \left|\frac{E}{1-\alpha}\right|^{1-\alpha} \leqslant M_\alpha\,. \tag{3.7}$$

Some consequences of this formula are

$$(3.8) \qquad |E| < M_0; \qquad l + \frac{1}{2} \leqslant K + \frac{1}{2} \leqslant \sqrt{M_1}; \qquad |E| \leqslant \frac{M_{\frac{1}{2}}^2}{4(l+\frac{1}{2})^2} .$$

The latter can be compared with the exact result for the Coulomb potential $V = -(M_{\frac{1}{2}}/x)$

$$|E| \leqslant \frac{M_{\frac{1}{2}}^2}{4(l+1)^2} .$$

Finally by integration of the identity

$$\psi D(E,\beta)\frac{\partial\psi}{\partial\beta} - \frac{\partial\psi}{\partial\beta} D(E,\beta)\psi = \frac{\psi^2}{x^2} + \frac{\mathrm{d}E}{\mathrm{d}\beta}\psi^2 ,$$

n the interval $0 \ldots \infty$ we find

$$3.\ 9) \qquad \frac{\mathrm{d}E}{\mathrm{d}\beta}\int_0^\infty \psi^2\,\mathrm{d}x = -\int_0^\infty \frac{\psi^2}{x^2}\,\mathrm{d}x .$$

Eq. (3.9) can be inserted in eq. (3.4) thus obtaining after some algebra

$$(3.10) \qquad P\left(\frac{\mathrm{d}\beta}{\mathrm{d}E}\right) \leqslant M_\alpha \frac{\mathrm{d}E}{\mathrm{d}\beta} .$$

If $\alpha = 0$ or $\alpha = 1$ this inequality can be integrated and the result is

$$(3.11) \qquad \begin{cases} |E(l')| < |E(l)| \dfrac{M_1 - (l'+\frac{1}{2})^2}{M_1 - (l+\frac{1}{2})^2} , & l' > l. \\ |E(l)| < M_0 - \dfrac{(l+\frac{1}{2})^2}{(l'+\frac{1}{2})^2}[M_0 - |E(l')|] . \end{cases}$$

4. – The shadow and resonance states.

We know from the results of ref. [1] that if β_0 is a pole of $S(\beta)$ there is a solution ψ of the Schrödinger equation

$$(4.1) \qquad D(E,\beta_0)\psi = 0 , \qquad E > 0,\ \mathrm{Im}\,\beta_0 < 0 ,$$

with the boundary conditions:

$$(4.2) \qquad \begin{cases} \psi \sim c x^{l_0+1} & x \to 0 , \qquad l_0 + \frac{1}{2} = \sqrt{-\beta^0}, \\ \psi \sim \exp[i\sqrt{E}x] & x \to \infty . \end{cases}$$

If $V(x)$ is of the form (2.3) we can continue it for complex values of x in the domain $\mathrm{Re}\,(x) \geqslant 0$. According to a general theorem (see ref. [5]) ψ can also be continued in the same region. Let us consider the function $\psi(iy) = \chi(y)$. χ satisfies the differential equation

$$D(E,\beta)\chi = \ddot{\chi} - E\chi + (\beta + \tfrac{1}{4})y^{-2}\chi + V(iy)\chi = 0\,, \tag{4.3}$$

where dots mean y-derivatives. From $\int \chi^*(D(E,\beta_0)\chi)\,\mathrm{d}y = 0$ by using the same techniques of Section **3** we find

$$\mathrm{Im}\,\beta \int_0^\infty |\chi|^2 \frac{\mathrm{d}y}{y^2} + \int_0^\infty \mathrm{d}y\,|\chi|^2\,\mathrm{Im}\,V(iy) = 0\,, \tag{4.4}$$

$$\int_0^\infty \left[\left| \dot{\chi} - \frac{\chi}{2y} \right|^2 + E|\chi|^2 - \mathrm{Re}\,\beta\,\frac{|\chi|^2}{y^2} - \mathrm{Re}\,(V(iy))\,|\chi|^2 \right] \mathrm{d}y = 0\,, \tag{4.4'}$$

$E - \mathrm{Re}\,\beta - \mathrm{Re}\,V$ must be negative somewhere in $0 \ldots \infty$.

In discussing these integral we must remember that for large positive y we have $\chi \sim \exp[-\sqrt{E}y]$. χ is therefore L^2.

If we assume now that $V(iy)$ satisfies the set of inequalities

$$\begin{cases} y^{2\alpha}|\,\mathrm{Im}\,V(iy)\,| < N'_\alpha\,; & y^{2\alpha}|\,\mathrm{Re}\,V(iy)\,| < N''_\alpha\,, \\ y^{2\alpha}|\,V(iy)\,| < N_\alpha\,; & N_\alpha < |\,N'_\alpha + iN''_\alpha\,|\,, \end{cases} \tag{4.5}$$

we can draw the following conclusions by using the same methods of Section **3**:

$$|\,\mathrm{Im}\,\beta\,| < N'_1; \qquad \left|\frac{E}{1-\alpha}\right|^{1-\alpha} \left|\frac{\mathrm{Re}\,\beta}{\alpha}\right|^{\alpha} \leqslant N''_\alpha \qquad \text{if} \quad \mathrm{Re}\,\beta < 0\,, \tag{4.6}$$

$$\text{no restrictions} \qquad \text{if} \quad \mathrm{Re}\,\beta > 0\,.$$

Condition (4.6) provides some bounds on L if N is finite. We have for instance $L \leqslant \mathrm{Re}\,\sqrt{N''_1 + iN'_1} - \tfrac{1}{2}$. The interesting case of Yukawa potential does not admit a finite N'_1 and we must therefore use a more sophysticated technique. To this purpose let us multiply (4.4) by $\cos\eta$, (4.4') by $\sin\eta$, and add, where $0 < \eta < \pi/2$. The result to be used only when $(\cos\eta\,\mathrm{Im}\,\beta + \sin\eta\,\mathrm{Re}\,\beta) \leqslant 0$ is

$$\left(\frac{E}{1-\alpha}\right)^{1-\alpha} \sin\eta^{1-\alpha} \left|\frac{\cos\eta\,\mathrm{Im}\,\beta + \sin\eta\,\mathrm{Re}\,\beta}{\alpha}\right|^{\alpha} \leqslant N_\alpha\,. \tag{4.7}$$

[5] E. T. Whittaker and G. N. Watson: *A Course of Modern Analysis* (Cambridge, 1952).

In the β plane cond. (4.5) is in general different from (4.6). If equality would hold in (4.5) this equation would represent a set of straight lines in the β plane. The permissible domain lies then below the real axis and above the envelope of these lines. For $\alpha = 0$ there is no envelope. For $\alpha = 1$ the envelope is the circle $|\beta| = N_1$, with $0 \leqslant \arg\sqrt{-\beta} \leqslant \pi/2$, and the line $\operatorname{Im}\beta = -N$. The corresponding conditions are $0 > \operatorname{Im}\beta > -N_1$ and $\operatorname{Re}\beta > -\sqrt{N^2 - \operatorname{Im}\beta^2}$. If $\alpha = \frac{1}{2}$ the envelope is most simply expressed in the plane as

$$\operatorname{Re} l < \frac{N_{\frac{1}{2}}}{2\sqrt{E}} - \frac{1}{2}. \tag{4.8}$$

This condition can be easily applied to the Yukawa potential. The other values of α require a much more involved algebra and provide a bound only if $\alpha > \frac{1}{2}$.

The same procedure can be applied to the resonance states. The resonance state can be defined as the solution of (4.1) with the boundary conditions (4.2) but with l real and integer and E complex. The branch of $\sqrt{E}$ has to be choosen in order to have $\operatorname{Im} E < 0$, this condition follows from the continuity equation. We find instead of (4.6)

$$\operatorname{Im} E < N_0'; \qquad \left|\frac{\operatorname{Re} E}{1-\alpha}\right|^{1-\alpha} \left|\frac{\beta}{\alpha}\right|^{\alpha} < N_x'', \qquad \text{if} \quad \operatorname{Re} E > 0, \tag{4.9}$$

otherwise no restriction.

And instead of (4.7)

$$\left|\frac{\beta \sin\eta}{\alpha}\right|^{\alpha} |\cos\eta \operatorname{Im} E + \sin\eta \operatorname{Re} E|^{1-\alpha} < N_\alpha. \tag{4.10}$$

The allowed areas for the resonances in the E plane are for $\alpha = 0$ the union of the circle $|E| < N_0$ and of the strip $\operatorname{Re} E < 0$. If $\alpha = \frac{1}{2}$ and $k^2 = E$ we have $\operatorname{Re} k < N_{\frac{1}{2}}/(2l+1)$.

5. – The physics of the shadow states.

It can be easily seen that shadow states are closely connected with resonances. Take namely the formula

$$\frac{\mathrm{d}E}{\mathrm{d}\beta} = -\frac{\int_0^\infty \psi^2 x^{-2} \mathrm{d}x}{\int_0^\infty \psi^2 \mathrm{d}x}. \tag{5.1}$$

Eq. (5.1) is a close analog of (3.9). Let we suppose now that $\operatorname{Im}\beta$ is very small. A small variation $\Delta\beta$ in β produces a small change ΔE in E. We choose $\Delta\beta = -i\operatorname{Im}\beta$ so that $\beta' = \Delta\beta + \beta$ is real. Correspondingly the energy will become complex $E + \Delta E = E + i\operatorname{Im}\beta/\langle R^2\rangle$, $1/\langle R^2\rangle$ being the average value of $1/x^2$ in the shadow state. These rough arguments show that there should be a resonance at the energy E if $\operatorname{Re} l$ is integer. The mean life of the resonance is given by $1/\Delta E$ or $\langle R^2\rangle/\operatorname{Im}\beta$ having adopted units such that $\hbar = c = 2m = 1$. What physical meaning can we associate with $\Delta\beta$? Classically a resonance is a scattering process where the path of the incoming particle winds up several times around the center of force and has therefore the opportunity of interacting for a long period with the potential. We can suppose that the radius of the orbit can be conveniently approximated by R. If l is the orbital momentum the angular speed $\mathrm{d}\varphi/\mathrm{d}t$ is $2l/\langle R^2\rangle$.

During the interaction time the particle travels the angle $\Delta\varphi = \Delta t \cdot \mathrm{d}\varphi/\mathrm{d}t = 1/\operatorname{Im} l$ before it is lost again in the scattering process. The above estimate can be also checked as follows. We write the total wave function as

$$\Psi = \frac{1}{r} Y_l^m(\theta, \varphi)\, \psi(r)\,. \tag{5.2}$$

Of course Ψ is not a true wave function since it is multivalued in the angles. However if we disregard these irregularities it provides a fairly accurate description of the scattering process when we are sufficiently far from the branch points of Y_l^m. Take now $m = l$. We have $Y_l^l = P_l^l(\theta)\exp[il\varphi]$. This function is multivalued in φ and $\operatorname{Im} l$ provides a damping factor in the orbit which will allow the particle to travel by an angle $\Delta\varphi$ of the same order of magnitude as found before. We can confidently assert that a shadow state with a very small $\operatorname{Im} l$ originates from a resonance in the energy. In writing up a Mandelstam representation of a given process the number of subtractions L has to be equal or larger than the angular momentum of the intermediate resonance. The above arguments lose their strength if $\operatorname{Im}\beta$ is no longer small because then the life-time of the resonance becomes too short.

An interesting interpretation of the shadow states, from which the name derives, follows from the work of LEVY and KELLER on the theory of diffraction (6). These authors have shown that in calculating the diffraction of a field against a smooth object it is possible to modify the ordinary geometrical optics, and obtain reliable results, by introducing new kind of rays, the diffracted rays, besides the optical ones.

The diffracted rays are produced by incident rays which are tangent to the surface of the body. A part of the incident ray, instead of continuing along

(6) B. LEVY and J. KELLER: *Diffraction on a smooth object.*

the geometrical path, follows a geodesics along the body, until it leaves eventually the body along a tangent. The amplitude of the part of the ray which undergoes this process decreases exponentially with the arc of geodesics travelled by the diffracted ray. Diffracted rays obviously penetrate into the geometrical shadow of the object. By analyzing the scattering amplitude in terms of complex angular momenta LEVY and KELLER prove that diffracted rays arise from the shadow states. In their case Im is essentially a angular penetration factor of the ray into the shadow, thereby justifying the name shadow state. Unfortunately there is no immediate generalization of Levy and Keller's theory of diffraction to the simple potential scattering. It has to be pointed out that the boundary condition for a hard core potential, corresponding to a mooth object, yield to somewhat simple analytic properties of $S(\beta)$, in fact in this case $S(\beta)$ is meromorphyc in β, there being no cut in $0 < \beta < \infty$.

In view of the considerable amount of work being done on phenomenological hard core potentials for nucleon-nucleon scattering it would be desirable to extend our results to more realistic cases.

* * *

The author wishes to thank Professor B. LEVY and Professor J. KELLER for an illuminating discussion. This work was partially supported by an AEC grant. The author is also particularly grateful to Prof. R. E. MARSHAK for his hospitality at the University of Rochester.

RIASSUNTO

In questo lavoro si estendono i risultati di una nota precedente dell'autore in cui si è fatto uso di momenti angolari complessi. In particolare vengono derivate ineguaglianze concernenti il numero di sottrazioni nelle regole di dispersione alla Mandelstam per vari tipi di potenziale tra cui il potenziale di Yukawa.

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

Heat Flush and Mobility of Electric Charges in Liquid Helium.

II. – Turbulent Flow.

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Istituto Nazionale di Fisica Nucleare - Sezione di Padova

(ricevuto il 6 Agosto 1960)

Summary. — The effect of a counterflow of normal and superfluids in liquid helium II on a beam of ions across a wide channel is studied. When a critical value of the heat current is reached a beam of negative ions is strongly affected. This is ascribed to turbulence in the superfluid component and to a strong interaction between the negative ions and the vorticity as vortex lines. The positive ions are instead almost unaffected. The different behaviour of the two types of ions is discussed in terms of their structures.

1. – Introduction.

In a previous paper [1], later indicated by I, an experiment was described in which some elementary charged particles were dragged in a wide channel by a stream of excitations, care being taken to keep the velocity of this excitation flow in the sub-critical regime. In this way it has been quantitatively shown that the charged particles participate in the normal fluid flow, quite in agreement with the Landau picture of liquid helium.

In this paper we investigate the behaviour of this drag under higher flow velocities, above a critical threshold, in a regime which has been now accepted as turbulent. Up to date the most extensive work concerning turbulence in

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(**) Present adress: Dept. of Physics, University of Tennessee, Knoxville, Tenn.
(1) G. Careri, F. Scaramuzzi and J. O. Thomson: *Nuovo Cimento*, **13**, 186 (1959).

a wide channel has been performed by VINEN (2); in the following we assume the reader to be familiar with VINEN's work.

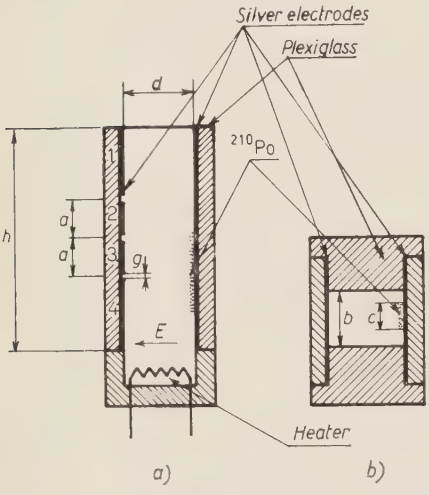

Fig. 1. – Schematic view of the apparatus: *a*) side view; *b*) top view. Dimensions are given in the text.

2. – Apparatus and technique.

The apparatus has been described in details in I. For the reader's comfort we will repeat here, in Fig. 1, the scheme of the apparatus. The layer of ^{210}Po emits α-particles that ionize intensely the helium. An electric field E is applied across the chamber which draws a small fraction of the ions of one sign from the ionized region toward the four collecting electrodes. Provision is made for measuring the current on each of these electrodes while the other three are grounded. A heater located at the bottom provides heat currents. Two apparatuses were used, whose nominal dimensions are (see Fig. 1):

	a	*b*	*c*	*d*	*g*	*h*	*A*
apparatus no. 1	5	5	2.5	3	0.1	60 mm	15 mm^2
apparatus no. 2	10	10	6	8	0.1	60 mm	80 mm^2

Apparatus no. 2 was also used in I.

In the first experiments a vacuum tube electrometer was used to measure the small ionization currents; later a vibrating reed electrometer was employed because of its greater sensitivity. By balancing out the input current to the electrometer with a fixed current of the opposite sign, very small changes in the current could be observed.

The method of measurement consisted in establishing a fixed temperature of the bath and a fixed constant electric field between the plates. In this condition, with a well insulated (greater than $10^{15}\,\Omega$) ceramic switch it was possible to measure, time by time, the currents on each of the four electrodes

(2) W. F. VINEN: *Proc. Roy. Soc.*, A **240**, 114, 128 (1957); A **242**, 493 (1957); A **243**, 400 (1957).

facing the ion source; in these conditions currents strong enough (10^{-13} A) were detectable on plates no. 3 and no. 4; a smaller current (10^{-11} A) on plate no. 2 and no current on plate no. 1. The « Heat Flush » measurements were performed in the following way: fixing the switch on one plate, a heat current $\dot{q}$ was passed through the channel and the eventual change in current Δi measured. Next the heat current was shut down and the original value of the current i_0—with no heat input—checked. Then a different heat current $\dot{q}'$ was produced by the heater and the cycle repeated. The same procedure was performed for each of the plates.

In the first runs, as the turbulent nature of the phenomenon had not yet been recognized, care was not taken to give ordered sets of heatings (*i.e.*, always increasing or always decreasing; so one part of the data shows the uncertainty due to hysteresis in the turbulence.

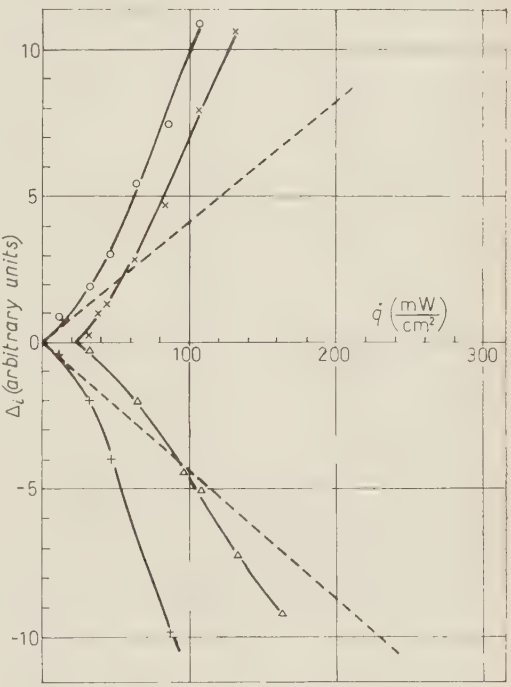

Fig. 2. – A typical plot of the change in current Δi versus heat input $\dot{q}$ in supercritical regime, for negative ions. The dotted lines show how the curves of plates no. 2 and no. 4 should be in non-turbulent regime. × plate no. 1; ○ plate no. 2; △ plate no. 3; + plate no. 4; $T = 1.185$ °K; $E = 120$ V/cm; apparatus no. 1.

3. – Experimental results.

In the description of the experimental results we will for simplicity speak of turbulence as if this were been already demonstrated to be the nature of the phenomena we are dealing with. In a later paragraph, however, we will justify this assumption.

With low heat currents it was possible to see with the apparatus no. 1 roughly the results described in I and obtained with the apparatus no. 2, namely a linear change in current with increasing heat input on plates no. 2 and no. 4 and no change on the other two plates. We say «roughly» because apparatus no. 1 was not sensitive enough for the measurements described in I, and it was not possible from the data of this apparatus to get the proper values of the mobility.

When the heat input is increased above a certain threshold value $\dot{q}^+$, which depends on the temperature, the linear dependence between Δi and $\dot{q}$ breaks down. The effect is not the same, however, for both positive and negative ions; in fact, the most striking feature of our measurements is the extremely different way in which the two kinds of ions behave when the transition into turbulent flow is made:

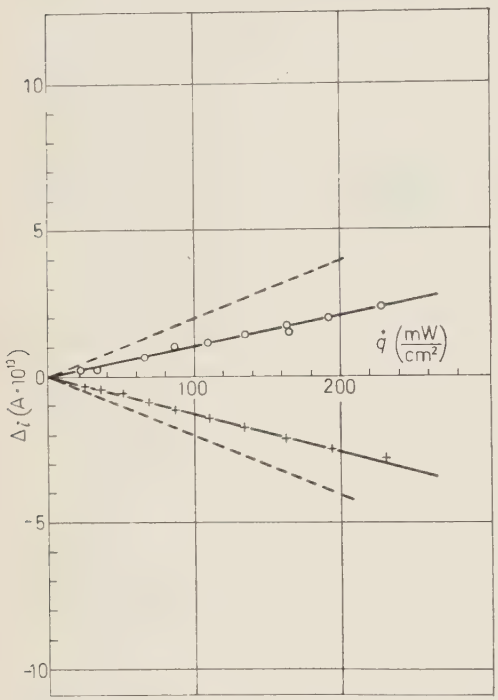

Fig. 3. – A typical plot of the change in current Δi versus heat input $\dot{q}$ in supercritical regime, for positive ions. The dotted lines show how the curves should be in non-turbulent regime. ○ plate no. 2; + plate no. 4; $T=1.185$ °K; $E=40$ V/cm; apparatus no. 1.

— negative ions experience a much stronger drag in the direction of flow of the heat current than in non-turbulent regime, increasing the slope s of the Δi versus $\dot{q}$ curve;

— positive ions, on the other hand, show only a small and discontinuous decrease of the slope.

Figs. 2 and 3 show the behaviour of the two kinds of ions at the same temperature. The dotted lines represent the way the curves should go in non-turbulent flow, calculated from the values of mobility measured by MEYER and REIF [3], CARERI, SCARAMUZZI and THOMSON [1], CARERI and CUNSOLO [4]. As negative ions show much stronger effects and, for this reason, have been more widely investigated, we will speak first and in more detail of them.

3·1. *Negative ions.*

A) One feature that put in evidence the turbulent nature of the flow above $\dot{q}^-$ was the fact that, when increasing heat inputs alternated with zero

[3] L. MEYER and R. REIF: *Phys. Rev.*, **110**, 279 (1958).
[4] G. CARERI and S. CUNSOLO: to be published in *Nuovo Cimento*.

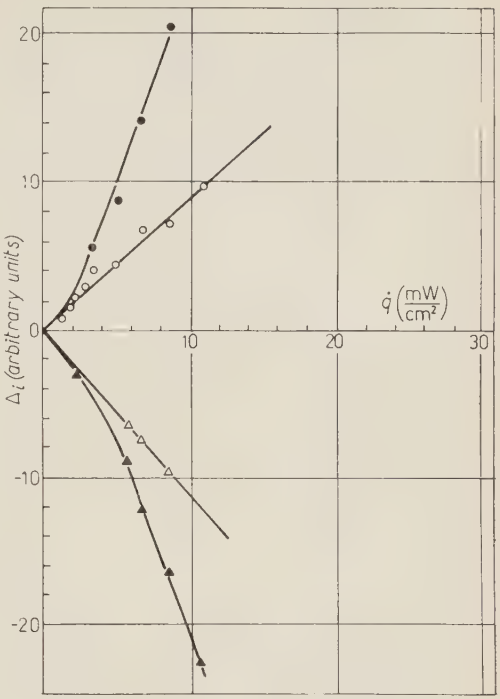

Fig. 4. – The change in current Δi versus heat input $\dot{q}$, for plates no. 2 and no. 4, for negative ions. The two successive values of Δi are due to the delay in setting up the turbulence. Plate no. 2: ○ Δi_1, ● Δi_2: Plate no. 4: △ Δi_1, ▲ Δi_2: $T=0.975$ °K; $E=125$ V/cm; apparatus no. 2.

heat input, once above the threshold value of $\dot{q}$, the values of Δi mesured on plates no. 2 and no. 4 were time dependant: *i.e.*, the current first changed by a certain amount Δi_1 and, after a certain time, the change increased steeply in absolute value, going to Δi_2, Δi_2 being always larger than Δi_1. Fig. 4 shows the two sets of points: notice that the Δi_1 are aligned with the origin of the co-ordinate system, as found in I for non-turbulent heat flush (in Fig. 2 only the Δi_2 points are reported). The measurement of the delay time was not easy and there was some uncertainty due to the long time constant of the electronics. However it was possible to conclude that:

— the delay time t decreased with increasing heat input; plotting $\log t$ versus $\log \dot{q}$, the empirical relation found by VINEN [2]

$$t = a\dot{q}^{-\frac{3}{2}}, \qquad (1)$$

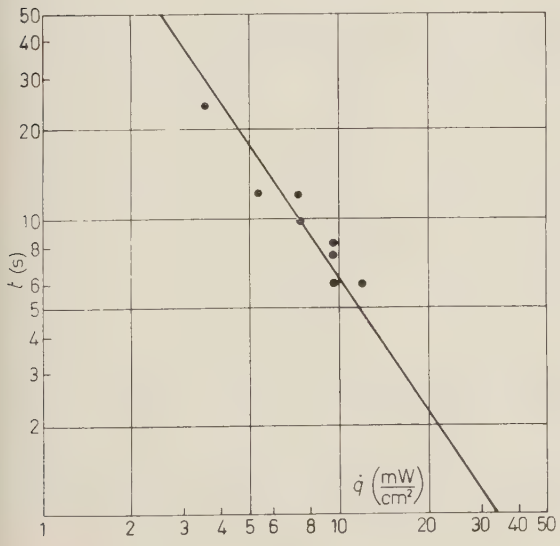

Fig. 5. – Plot of the log of the delay time t versus the log of the heat input $\dot{q}$. The full line represents eq. (1). The experimental points have been obtained on plate no. 2, with negative ions. $T=0.975$ °K; $E=38$ V/cm; apparatus no. 2.

was approximately verified. Fig. 5 shows our results: the full line is equation (1). Further, the value of a, deduced from the plot of Fig. 5, is a resonable extrapolation of the values found by VINEN: Fig. 6;

— for a given heat input the delay time depended on the hystory of the liquid; mainly on how long the heat current had been turned off before the trial in question.

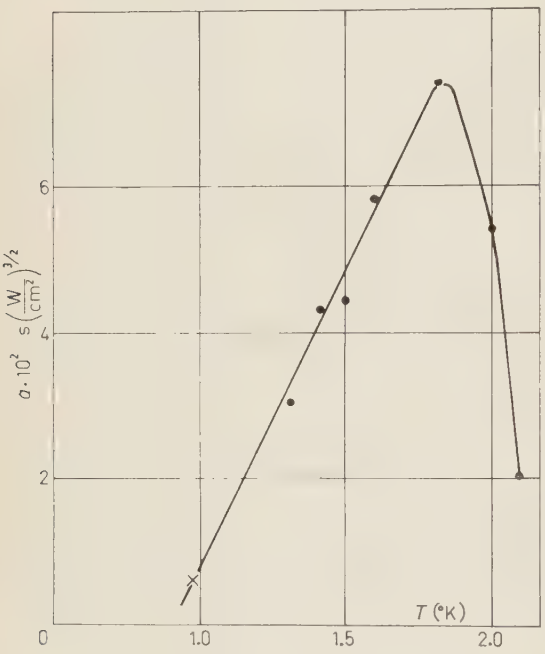

Fig. 6. – The constant a of eq. (1) as a function of temperature. ● Vinen's experimental points; ˣ our experimental point; $T = 0.975$ °K; $E = 38$ V/cm; apparatus no. 2.

These facts tend to show that we are dealing with turbulence, as VINEN also proposed to explain his results ([2]). With our delay time data it was not possible to determine the turbulence threshold $\dot{q}^+$ in a satisfactory way, using the method suggested by VINEN ([2]), the threshold being the limit to which $\dot{q}$ tends when the delay time t tends to infinity.

B) Another striking phenomenon needs to better investigated: the anomalous Δi versus $\dot{q}$ plots shown in Fig. 2. It is interesting to note that the deviation from the normal behaviour appears for all four plates in a different way, but almost at the same value of $\dot{q}$. Once this threshold is reached the behaviour of the plates is as follows:

a) plate no. 1 which, because of its position (Fig. 1), normally collects no current, experiences a current increasing with the heat input;

b) plate no. 2 shows a faster increase in current than in the non-turbulent regime;

c) plate no. 3, which would show no change in current for laminar flow of these velocities (see Fig. 1), shows a decrease with increasing heat input;

d) plate no. 4 shows a decrease larger than in the non-turbulent regime.

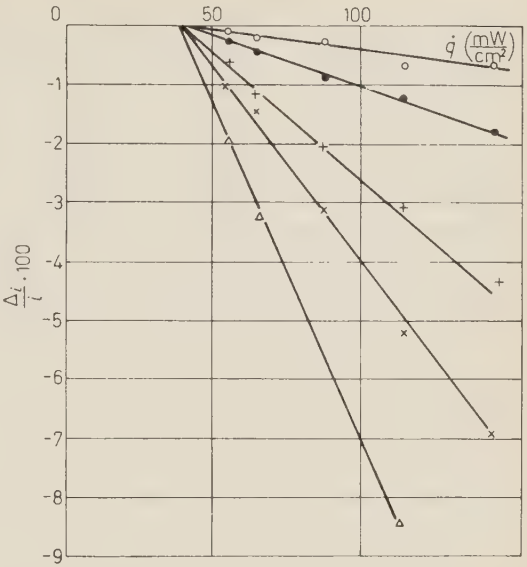

Fig. 7. – The change in total current, in percentage, $(\Delta i/i)\cdot 100$, versus heat input $\dot{q}$, for negative ions, with different applied electric fields E. ○ $E=100$ V/cm; ● $E=33$ V/cm; + $E=17$ V/cm; × $E=10$ V/cm; △ $E=3.3$ V/cm; $T=1.60$ °K; apparatus no. 1.

The values of $\dot{q}$ though not sharply defined appear to be the same for all four plates.

C) The change in current on plate no. 3 suggested the possibility of a change Δi_t in the total current received by all the plates. To test this possibility a few runs were performed in which all four plates were connected together to the electrometers. Fig. 7 shows the results of these experiments: a very strong decrease Δi_t was noticeable with a linear dependence on heat input, starting at a well defined value of $\dot{q}$ in good agreement with Vinen's values. It should be emphasized that similar graphs for different values of the applied field give the same critical heat current and that, plotting $\Delta i/i$ versus $1/E$, a linear dependance is found (Fig. 8).

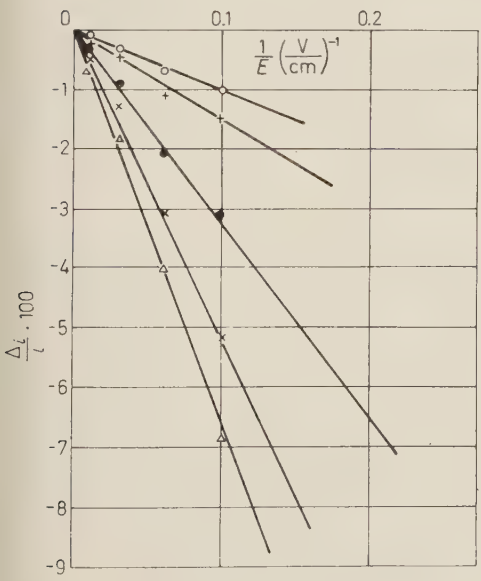

Fig. 8. – The change in total current in percentage, $(\Delta i/i)\cdot 100$, versus $1/E$ for negative ions, with different heat inputs $\dot{q}$. ○ $\dot{q}=54$ mW/cm²; + $\dot{q}=64$ mW/cm²; ● $\dot{q}=88$ mW/cm²; × $\dot{q}=114$ mW/cm²; △ $\dot{q}=145$ mW/cm²; $T=1.60$ °K; apparatus no. 1.

3·2. *Positive ions.* – The experimental data regarding positive ions are very poor, due mainly to the fact that, as the negative ions appeared to be a more sensitive tool to investigate turbulence, the last and more precise runs were dedicated

only to them. So we have at present few and uncertain data on the positive ions. However we can summarize what we know about them as follows:

— there is a small change in the slope of the Δi versus $\dot{q}$ curves for plates no. 2 and no. 4, in the opposite direction of that experienced with negative ions. This change takes place within a small range of $\dot{q}$'s near the value for the threshold of turbulence (Fig. 3);

— there is no change in current on plates no. 1 and no. 3;

— the total current doesn't change, even for very strong heatings.

4. – Discussion.

4·1. *Negative ions.* – Summarizing the phenomena of the previous section, we conclude as follows:

— from paragraphs *a*), *b*) and *d*) it follows that in turbulent flow the ions are dragged in the direction of the heat current much more than in the non-turbulent region;

— from paragraphs *c*) and *C*) it is seen that the current density, on the collecting electrodes, decreases.

All this can be understood, qualitatively at least, in terms of a decrease of mobility. In fact, if the mobility decreases, the ions need more time to cross the channel and, as the normal fluid velocity per unit heat input v_n^0 remains practically (*) unaffected by turbulence, the ions will be carried farther in the direction of the heat flow. A decrease of the mobility explains furthermore the change in current on plate no. 3 and the change in total current experienced with the four plates connected in parallel. A quantitative explanation of this behaviour requires the knowledge of unknown elements, such as the dependence of ion number density on mobility.

4·2. *Positive ions.* – The behaviour of positive ions consists of a sudden decrease in the slope of the Δi versus $\dot{q}$ plots for plates no. 2 and no. 4. The slope is given by (see I)

$$s = \frac{\Delta_i}{\dot{q}} = \frac{di_3}{aE} \frac{v_n^0}{\mu},$$

d, a – linear dimensions of the channel (see Fig. 1); E = applied electric field; i_3 = electric current (constant) on plate no. 3; v_n^0 = velocity of the normal

(*) See « Positive ions » soon later.

fluid per unit heat current density; μ = mobility. As it does not seam reasonable to impute the decrease in slope to an increase of the mobility, the only way to interpret this phenomenon is as a decrease of v_n^0. This could be explained in two ways:

— In classical fluids when the flow in a channel passes from a laminar to a turbulent type of flow, the velocity profile across the channel becomes much flatter, *i.e.* there is a decrease in the velocity in the center of the channel. For our geometry (apparatus no. 1) calculation gives a correction factor of 1.37 for the velocity. This is very close to what is observed (see Fig. 3), but this may be somewhat coincidental because we are dealing presumably with turbulence in the superfluid and what we notice is a phenomenon connected with the normal fluid. The explanation becomes reasonable if we suppose a strong interaction between the normal and superfluid via the vorticity.

— The second possibility arises from the expression from which we calculate v_n^0 (see I)

$$v_n^0 = \frac{1}{\varrho S T}, \tag{2}$$

ϱ = density of helium; T = temperature; S = entropy per unit mass. A decrease of v_n^0 could be explained by an increase of ϱ, T or S; we can discard changes in ϱ and elementary calculations give absolutely negligible T changes. The only possibility remains a change in S: that due to the small change in T can also be discarded by elementary calculations, in spite of the strong dependence of S on T. It remains to doubt the applicability of equation (2), because now part of the heat input is transformed into vorticity of the superfluid, due to the building up of a tangled mass of vortex lines. At the present state of the knowledge, this possibility cannot be further discussed in a quantitative way.

The change of v_n^0 here postulated should happen also for negative ions, but it is overcome by the phenomena imputed to the change of mobility.

5. – Concluding remarks.

While there is no doubt that negative charges can be used as tracers to detect turbulence in liquid helium, one would like to understand more what happens to them in this process. Actually it is quite reasonable that they should interact with vortex lines, because the free charges might tend to take up a position in the cores of the vortex lines to reduce their zero point energy ([5]). This is quite in agreement with a discussion on the nature of the

([5]) L. ONSAGER: private communication.

ionic structures made by CARERI, FASOLI and GAETA ([6,7]): in this paper the positive ion is described as a droplet of polarized helium atoms around the positive charge, while the negative one is thought as an empty bubble where the free electron is self-trapped by a shell of polarized atoms. According to these structures only the free electron, because of its so light mass, can substancially lower its zero-point energy by setting himself in the empty core of a vortex line, while the positive ion, being so massive, will remain essentially unaffected. This interaction term of the negative ion with the vortex line will give rise to a large collision cross-section and finally to a decrease of the mobility of the charges.

To test the consistency of the above picture it seems that this work should be extended and substanciated in the following ways:

— By direct measurement of the mobility of the negative charges in a known array of vortical lines, like those that should exist in rotating helium, according to FEYNMAN ([8]), one could determine the scattering cross-section.

— Using different beams of negative particles in different places, along a channel, one should be able to say more about the propagation rate of the turbulence in a wide channel; work along this line is in progress in this laboratory ([9]).

— By careful inspection of the drag velocity of the positive ions in different places across a channel, one should obtain the velocity profile of the normal fluid, both in subcritical and supercritical conditions.

([6]) G. CARERI, U. FASOLI and F. S. GAETA: *Nuovo Cimento*, **15**, 774 (1960).

([7]) For a general review of this subject see also: G. CARERI in GORTER'S: *Progress in Low Temperature Physics*, Vol. 3° (Amsterdam), to be published shortly.

([8]) R. P. FEYNMAN: *Progress in Low Temperature Physics*, **1**, 17 (1955).

([9]) G. CARERI, W. D. MCCORMICK and F. SCARAMUZZI: submitted to the *Intern. Conf. on Low Temperature Physics* (Toronto, 1960).

RIASSUNTO

È stato studiato l'effetto di un controflusso di fluido normale e di superfluido in elio liquido II su un fascio di ioni in un canale largo. Quando si raggiunge un valore critico della corrente di calore un fascio di ioni negativi ne è fortemente affetto. Questo è imputato a turbolenza nel superfluido e ad una forte interazione tra ioni negativi e linee vorticali. Gli ioni positivi risentono molto meno il cambiamento di regime. Si discute il diverso comportamento dei due tipi di ioni in termini della loro struttura.

Sulla stabilità del moto di cariche elettriche in un campo di Biot e Savart.

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Istituto di Meccanica Razionale dell'Università - Napoli
Scuola di Perfezionamento in Fisica Teorica e Nucleare dell'Università - Napoli

(ricevuto l'8 Agosto 1960)

Riassunto. — Viene compiuta con metodi rigorosi un'analisi completa della stabilità del movimento di cariche elettriche in un campo di Biot e Savart, caso già parzialmente esaminato da Caianiello e Gatti, come esempio tipico dell'interesse che i campi magnetici prodotti da correnti d'altissima intensità potranno avere nella costruzione di acceleratori di particelle e in problemi di confinamento di ioni accelerati. Le conclusioni mostrano, tra l'altro, l'inattendibilità dei risultati che in siffatte questioni vengono spesso ottenuti con l'impiego di metodi approssimati.

1. – Introduzione.

In un recente lavoro (1) E. R. Caianiello e G. Gatti hanno fornito alcune ragioni favorevoli all'idea d'impiegare il campo magnetico prodotto da una corrente rettilinea nella costruzione di acceleratori di particelle. Precisamente nello studio di particolari movimenti di un corpuscolo elettrizzato in un campo di Biot e Savart, essi sono pervenuti alla conclusione che il moto della proiezione Q di P su un piano normale al conduttore, innanzi tutto, si svolge in una regione limitata che si può convenientemente circoscrivere pur d'impiegare correnti di grande intensità (dell'ordine di 10^5 A per energie di 10 MeV), e, in secondo luogo, esso verifica la condizione di essere un moto stabile.

Ma proprio la parte del succitato lavoro riguardante la stabilità del moto non può considerarsi soddisfacente, sia per l'adozione del metodo

(1) E. R. Caianiello e G. Gatti: *Nuovo Cimento*, **12**, 469 (1959).

delle piccole oscillazioni che, a stretto rigore, conduce a condizioni necessarie ma non in generale sufficienti per la stabilità, sia per altre non chiaramente giustificate approssimazioni compiute nell'applicazione del metodo stesso.

Il Prof. CAIANIELLO mi ha perciò suggerito di riesaminare da un punto di vista rigoroso l'intera questione della stabilità, ciò che appunto costituisce l'oggetto del presente articolo ove più generalmente l'analisi della stabilità viene estesa a *tutti* i movimenti della particella.

Innanzi tutto, con l'ausilio di una nota discussione di WEIERSTRASS [2], si osserva (Sezione 2'2) che, qualunque sia il sistema di condizioni iniziali, Q si mantiene *sempre* in regioni limitate e che per esso si presentano soltanto i seguenti casi:

1) oscillazioni su segmenti rettilinei;

2) movimenti circolari uniformi;

3) la traiettoria si svolge in una corona circolare (col centro sul conduttore e con raggi dipendenti dalle condizioni iniziali) toccando alternativamente l'una o l'altra delle circonferenze che delimitano la corona, in maniera che sia sempre la stessa la differenza di anomalia di due punti di contatto consecutivi.

La stabilità (*ridotta* al moto di Q) è stata studiata nelle Sezioni 3'1, 3'2, 3'3 col risultato che *essa non si presenta in alcun caso*, contrariamente alle conclusioni della trattazione approssimata di Caianiello e Gatti. Sorvolando sui casi 1) e 2), per i quali la dimostrazione si consegue facilmente, per la trattazione del caso 3) viene introdotta alla Sezione 3'1 una classe C di movimenti σ dipendenti con continuità da un parametro λ e comprendente il moto σ^* in esame. È proprio rispetto a C che si dimostra l'instabilità di σ^*, provando (Sezione 3.2) che in corrispondenza ad infiniti movimenti σ di C, arbitrariamente prossimi a σ^* nell'istante iniziale, la distanza tra le posizioni sincrone del punto mobile in σ, σ^*, non si mantiene al variare del tempo sempre inferiore ad un limite prefissato.

Stante questa instabilità, acquista allora interesse, in ordine all'idea di utilizzare il campo di Biot e Savart nel modo anzidetto, il problema d'individuare, almeno, qualche *stabile* aspetto significativo del moto di Q. Ciò appunto costituisce l'argomento delle Sezioni 3'4, 3'5, in cui si perviene alle proposizioni seguenti:

I) *Per ogni moto circolare vi è stabilità ridotta all'orbita l^*, nel senso che la distanza da l^* del punto mobile Q in un generico moto σ inizialmente abbastanza prossimo a σ^*, si conserva arbitrariamente piccola.*

[2] T. LEVI CIVITA e U. AMALDI: *Lezioni di meccanica razionale* (Bologna, 1951), cap. I, § 6.

II) *Nel caso* 3) *i raggi della corona in cui si svolge la traiettoria del moto perturbato* σ *sono arbitrariamente poco discosti dai corrispondenti relativi al moto imperturbato* σ^*, *sempre che, nell'istante iniziale,* σ *sia sufficientemente vicino a* σ^* (*anche se la stabilità ridotta all'orbita* l^* *sussiste solo se* l^* *non è chiusa*).

2. – Determinazione dei movimenti di un corpuscolo elettrizzato in un campo di Biot e Savart.

2'1. – Sia P un corpuscolo elettrizzato di carica q e di massa (relativistica) m, mobile nel campo magnetico prodotto da una corrente rettilinea d'intensità costante i.

Con riferimento ad un sistema di coordinate cilindriche r, θ, z, in cui l'asse z coincide con l'asse del conduttore, le equazioni del moto di P ammettono gli integrali primi

$$(1)\qquad \begin{cases} \dot r^2 + r^2\dot\theta^2 + \left(K\log\dfrac{r}{r_0} + \dot z_0\right)^2 = v^2\,, \\ r^2\dot\theta = a\,, \\ \dot z = K\log\dfrac{r}{r_0} + \dot z_0\,, \end{cases}$$

avendo posto

$$K = h\,\frac{iq}{m}\,,$$

(con h costante dipendente dalla scelta delle unità di misura), ed in cui $r_0, \dot z_0$ denotano i valori di r, $\dot z$ relativi all'istante iniziale, mentre v, a sono due costanti di cui la prima è ovviamente la grandezza della velocità di P.

Il moto della proiezione Q di P su un generico piano π, normale a z, coincide col moto piano di un punto soggetto ad una particolare forza centrale avente per centro il punto O in cui z interseca π; le (1_1), (1_2) sono appunto gl'integrali, rispettivamente delle forze vive e delle aree, delle equazioni di codesto moto.

Ciò premesso, non è difficile riconoscere che le (1) medesime possono assumersi come le equazioni del moto di P, salvo però escludere il caso di orbite circolari di Q in cui, come è noto, gli integrali delle forze vive e delle aree non equivalgono perfettamente alle equazioni del moto di un punto soggetto a forza centrale (3).

L'integrazione del sistema delle (1_1), (1_2) introdurrà come costante arbitraria il valore iniziale θ_0 di θ e, poichè sono arbitrarie anche le costanti r_0, $\dot z_0$,

(3) Cfr. loc. cit. in (2), cap. II, § 2, n. 4.

r, a, si riconosce che il moto di Q viene a dipendere da cinque costanti arbitrarie. La successiva integrazione della (1_3) introdurrà una nuova costante e si otterranno così in tutto le sei costanti arbitrarie da cui dipende l'integrale generale di ogni problema di moto di un punto libero.

2'2. – In merito al problema di determinare il moto di Q, cominciamo con l'esaminare il caso eccezionale $a = 0$. Esclusa l'ipotesi che Q stia in quiete in O, il moto di Q sarà rettilineo lungo una retta passante per O ed a questa corrisponderà un moto di P in un piano contenente z.

La legge oraria potrà determinarsi ricorrendo alla (1_1) che, per essere $\dot{\theta} = 0$, diventa

$$\dot{r}^2 = v^2 - \left(K \log \frac{r}{r_0} + \dot{z}_0\right)^2,$$

e che, posto

$$f(r) = v^2 - \left(K \log \frac{r}{r_0} + \dot{z}_0\right)^2,$$

prende la nota forma

$$\dot{r}^2 = f(r)\,, \tag{2}$$

alla quale sono applicabili i risultati della citata discussione di Weierstrass [4].

La condizione di realtà del moto $f(r) \geqslant 0$ impone che

$$-v \leqslant K \log \frac{r}{r_0} + \dot{z}_0 \leqslant v\,,$$

e con ciò si escludono sia il caso di moti a meta infinitamente lontana $(r \to \infty)$ e quindi che Q nel suo moto non incontri alcuna radice di $f(r)$, sia il caso che Q transiti per O $(r = 0)$ o, anche, che esso si avvicini indefinitamente ad O $(r \to 0)$. In quanto poi alle due uniche radici di $f(r)$, escluso com'è naturale che sia $r = 0$ o $K = 0$, esse sono semplici; infatti è

$$\frac{\mathrm{d}f}{\mathrm{d}r} = -2\left(K \log \frac{r}{r_0} + \dot{z}_0\right)\frac{K}{r}\,,$$

mentre che in corrispondenza alla sopradette radici risulta

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \mp\frac{2vK}{r}\,.$$

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[4] Cfr. loc. cit. in [2].

Possiamo pertanto concludere che quando $a = 0$, gli unici movimenti possibili sono quelli in cui Q descrive, su rette passanti per O, segmenti non contenenti O, oscillando periodicamente tra gli estremi di questi.

Supposto ora $a \neq 0$ e quindi, per la (1_2), $\dot{\theta} \neq 0$ durante tutto il moto, si perviene all'equazione differenziale della traiettoria di Q eliminando dalla (1_1) il tempo ed assumendo come variabile indipendente θ. Precisamente, posto

$$u = \frac{1}{r}\,,\quad \alpha = \frac{v}{a}\,,\quad \beta = -\frac{K}{a}\,,\quad \gamma = \frac{\dot{z}_0 - K\log r_0}{a}\,, \tag{3}$$

$$F(u) = \alpha^2 - u^2 - (\beta \log u + \gamma)^2\,, \tag{4}$$

si ha l'equazione

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 = F(u)\,, \tag{5}$$

che ha la stessa forma della (2).

Innanzi tutto, per essere

$$\lim_{u\to 0} F(u) = -\infty\,,$$

ogni soluzione reale $u(\theta)$ della (5), dovendo invece verificare la condizione

$$F(u) \geqslant 0\,, \tag{6}$$

ha estremo inferiore *positivo*. Ciò comporta che il raggio vettore $r = 1/u$ sia una funzione limitata di θ, ovvero che la corrispondente traiettoria di Q si svolga in una regione limitata del piano π.

Sempre per la (6), inoltre, deve essere $r \geqslant 1/\alpha$ e questo implica che Q non solo non possa transitare per O, ma che in più esso non possa avere O come meta asintotica.

Si riconosce [5] allora che relativamente alla traiettoria di Q rimangono come possibili a priori soltanto i seguenti casi: a) *orbite circolari*; b) *tendenza asintotica di $u(\theta)$ verso un valore finito e non nullo*; c) *oscillazioni periodiche di $u(\theta)$ tra due valori estremi.*

Il verificarsi del caso a) dipende dall'esistenza di radici multiple di $F(u)$: esattamente, se si può disporre dei valori iniziali in maniera che risulti $F(1/r_0) = F'(1/r_0) = 0$, l'orbita del corrispondente moto di Q è circolare con centro in O e raggio r_0. Con l'ausilio delle (3), (4) e delle relazioni che dalle (1_1), (1_2)

[5] Cfr. loc. cit. in [2].

traggonsi tra le costanti r, a ed i dati iniziali, si controlla agevolmente che tali condizioni equivalgono alle seguenti altre

$$\dot{r}_0 = 0\,, \qquad K\dot{z}_0 > 0\,, \qquad |\dot{\theta}_0| r_0 = \sqrt{K\dot{z}_0}\,, \tag{7}$$

che mostrano la possibilità effettiva di orbite circolari (anzi per la (1_2) di moti circolari *uniformi* di Q, ovvero, ancora, per la (1_3) di moti elicoidali uniformi di P).

Il caso *b*) invece, qualunque sia il sistema di condizioni iniziali, non può presentarsi.

Difatti, supposto che vi sia una soluzione reale $u(\theta)$ della (5) tale che al divergere di θ, $u(\theta)$ si avvicini asintoticamente ad un valore $\overline{u}$, questo dovrà essere radice multipla di $F(\overline{u})$. Ma è

$$F'(u) = -2\left[u + \frac{\beta}{u}(\beta \log u + \gamma)\right],$$

$$F''(u) = -2\left\{1 + \frac{\beta}{u^2}[\beta - (\beta \log u + \gamma)]\right\},$$

per cui, avendosi $F'(\overline{u}) = 0$, risulta

$$F''(\overline{u}) = -2\left(2 + \frac{\beta^2}{\overline{u}^2}\right) < 0\,.$$

Cosicchè, per $u = \overline{u}$, $F(u)$ ha un massimo relativo proprio e tale massimo è nullo per essere $\overline{u}$ radice di $F(u)$. Pertanto per $u(\theta)$ abbastanza prossima ad $\overline{u}$, ma distinta da $\overline{u}$, si ha $F(\overline{u}) < 0$, in contrasto con la (6).

Onde, quando $a \neq 0$, oltre il caso di orbite circolari, effettivamente si verifica soltanto il caso *c*) in cui $u(\theta)$ oscilla tra due valori estremi u_1, u_2, dipendenti dalle condizioni iniziali, e che sono radici semplici di $F(u)$. La corrispondente traiettoria si svolge nella corona circolare delimitata dalle circonferenze concentriche in O e di raggi $1/u_1$, $1/u_2$, toccando alternativamente l'una o l'altra di queste in modo che la differenza tra i valori di θ relativi a due punti di contatto consecutivi (*angolo apsidale*) sia sempre la stessa e precisamente data, se si suppone $u_1 < u_2$ da

$$\Theta = \int_{u_1}^{u_2} \frac{\mathrm{d}u}{\sqrt{F(u)}}\,. \tag{8}$$

Se Θ è commensurabile con π, l'orbita è chiusa; nel caso opposto essa si avvolge infinite volte attorno ad O e riempie la predetta corona nel senso che comunque si scelga in questa un punto, l'orbita finisce col passare ad una

distanza da questo piccola ad arbitrio (6). Come apparirà in seguito *entrambe* queste circostanze si presentano, in dipendenza del sistema di condizioni iniziali (7).

3. – Questioni di stabilità del movimento.

3'1. – Il più generale moto di Q dipende da cinque costanti arbitrarie (Sezione 2'1) e come tali potremo assumere i valori iniziali r_0, θ_0, $\dot{r}_0$, $\dot{\theta}_0$, $\dot{z}_0$. La funzione $F(u)$, oltre che da u dipende appunto attraverso le costanti α, β, γ da tali valori ed esplicitando questa dipendenza mediante un ricorso alle (3) ed alle già richiamate relazioni che dalle (1_1), (1_2) si deducono tra le costanti v, a ed i dati iniziali, si ha precisamente

$$F(u) = \frac{1}{r_0^2} - u^2 + \frac{1}{r_0^4 \dot{\theta}_0^2}\left[\dot{r}_0^2 - K^2 \log^2 (r_0 u) + 2K\dot{z}_0 \log (r_0 u)\right]. \tag{9}$$

Ciò premesso, vogliamo ora prendere in considerazione i movimenti di Q dal punto di vista della loro stabilità e cominceremo con l'analisi di un generico moto σ^* in cui $u(\theta)$ oscilli tra due radici semplici u_1, u_2 di $F(u)$. I casi che così si vengono ad escludere, verranno esaminati per ultimo.

Convenendo di contrassegnare con un asterisco ogni ente che si riferisca a σ^*, si ha innanzi tutto

$$F^*(u_1) = F^*(u_2) = 0\,. \tag{10}$$

Accanto a σ^* consideriamo poi un generico movimento σ tale che

$$r_0 = r_0^*, \qquad \theta_0 = \theta_0^*, \qquad \dot{\theta}_0 = \lambda \dot{\theta}_0^*, \tag{11}$$

con λ costante positiva e non superiore ad 1; in quanto agli ulteriori valori iniziali $\dot{r}_0$, $\dot{z}_0$, mostreremo che essi si possono scegliere in guisa che, analogamente a ciò che si ha per $F^*(u)$, sia

$$F(u_1) = F(u_2) = 0\,. \tag{12}$$

Calcolando difatti $F(u_1)$ e $F(u_2)$ mediante la (9) e tenendo conto delle (10), (11), le (12) si scrivono

$$\begin{cases} \dot{r}_0^2 + 2K\dot{z}_0 \log (r_0 u_1) = \lambda^2 \dot{r}_0^{*2} - K^2(\lambda^2 - 1) \log^2 (r_0 u_1) + 2\lambda^2 \dot{z}_0^* K \log (r_0 u_1)\,, \\ \dot{r}_0^2 + 2K\dot{z}_0 \log (r_0 u_2) = \lambda^2 \dot{r}_0^{*2} - K^2(\lambda^2 - 1) \log^2 (r_0 u_2) + 2\lambda^2 \dot{z}^* K \log (r_0 u_2)\,, \end{cases}$$

(6) Cfr. loc. cit. in (2), cap. II, § 2, n. 9.

(7) Ciò non è affatto assicurato a priori; nel caso delle forze elastiche, ad es., l'angolo apsidale è indipendente dalle condizioni iniziali.

e questo sistema considerato nelle incognite $\dot{r}_0^2$, $\dot{z}_0$, per essere $u_1 \neq u_2$, ammette come unica soluzione

$$(13) \qquad \begin{cases} \dot{r}_0^2 = \lambda^2 \dot{r}_0^{*2} + K^2(\lambda^2 - 1) \log(r_0 u_1) \log(r_0 u_2) , \\ \dot{z}_0 = \lambda^2 \dot{z}_0^* - \dfrac{K}{2}(\lambda^2 - 1)\left[\log(r_0 u_1) + \log(r_0 u_2)\right] ; \end{cases}$$

si riconosce poi che i due valori forniti dalla (13_1) per $\dot{r}_0$ sono reali e ciò in virtù delle limitazioni poste su λ e per essere ovviamente $\log(r_0 u_1) \log(r_0 u_2) \leqslant 0$.

Completeremo infine la definizione di σ convenendo ulteriormente di assumere per $\dot{r}_0$ la radice che ha il segno di $\dot{r}_0^*$, salvo quando $\dot{r}_0^* = 0$ nel qual caso intenderemo prefissato ad arbitrio il segno di $\dot{r}_0$.

Si viene con ciò, in corrispondenza ai valori di λ nell'intervallo (0,1), a definire una classe C di movimenti di Q, ciascuno caratterizzato mediante i valori r_0, θ_0, $\dot{r}_0$, $\dot{\theta}_0$, $\dot{z}_0$, dati dalle (11), (13) come funzioni continue di λ e tendenti ordinatamente, quando λ si approssima ad 1, ai valori iniziali relativi a σ^*.

Pertanto, per il generico moto σ di C, potremo scrivere

$$r = r(\lambda, t) , \qquad \theta = \theta(\lambda, t) ,$$

i secondi membri riducendosi a $r^*(t)$, $\theta^*(t)$ per $\lambda = 1$.

Ebbene noi proveremo che σ^* è instabile rispetto ai movimenti σ di C e ciò è sufficiente per dire che σ^* è instabile senz'altro, mentre invece un'asserzione di stabilità avrebbe richiesto un confronto di σ^* con *tutti* i movimenti di Q ad esso inizialmente prossimi.

Esattamente dimostreremo che *è possibile determinare due numeri positivi* ε_1, ε_2, *con la condizione che comunque si scelga il numero* η *positivo e minore di* 1, *esistono nell'interrallo* $(\eta, 1)$ *aperto a destra, infiniti valori di* λ *tali che le relazioni*

$$|r(\lambda, t) - r^*(t)| < \varepsilon_1 , \qquad |\theta(\lambda, t) - \theta^*(\lambda)| < \varepsilon_2 ,$$

non siano verificate simultaneamente per qualunque t.

3'2. – Allo scopo di dimostrare il teorema ora enunciato, premettiamo le seguenti due osservazioni:

α) *in ognuno dei movimenti* σ *di* C *si hanno per* u *oscillazioni periodiche tra gli stessi estremi* u_1, u_2, *in cui* u *oscilla in* σ^*;

β) *comunque si scelga* η *positivo e minore di* 1, *si può determinare* λ *verificante la condizione* $\eta \leqslant \lambda < 1$ *e tale che risultino tra di loro incommensurabili gli angoli apsidali* Θ *e* Θ^* *relativi a* σ, σ^*.

Innanzi tutto, indicando con $F(u, \lambda)$ la medesima $F(u)$ considerata, per i

moti σ di C nella sua ulteriore dipendenza da λ, in virtù delle (9), (11), (13) si ha facilmente

$$F(u, \lambda) - F^*(u) = \frac{K^2}{\lambda r_0^4 \dot{\theta}_0^{*2}} (\lambda^2 - 1) \log \frac{u}{u_1} \log \frac{u}{u_2},$$

da cui si vede che per ogni λ *interno* all'intervallo (0,1) risulta

$$F(u, \lambda) \geqslant F^*(u), \tag{14}$$

e quindi

$$F(u, \lambda) \geqslant 0, \tag{15}$$

al variare comunque di u in (u_1, u_2), sussistendo inoltre il segno di uguaglianza in ciascuna di queste relazioni se e solo se $u = u_1$, $u = u_2$. Poichè, d'altra parte, per la (11_1) il valore iniziale di u in ognuno dei moti σ di C appartiene all'intervallo (u_1, u_2), per la dimostrazione dell'osservazione α), rimane da far vedere che u_1, u_2, sono radici semplici anche per $F(u, \lambda)$. Ma ciò è evidente giacchè se ad es. u_1 fosse radice multipla di $F(u, \lambda)$, tale funzione (Sezione 2'2) in un intorno *completo* di u_1, privato del solo valore u_1, dovrebbe assumere valori negativi, in contrasto con la (15).

La β), invece, è conseguenza del fatto che come subito proveremo *la* (8) *esprime, per i movimenti della classe C in esame, l'angolo apsidale come funzione* $\Theta(\lambda)$ *continua quando* $0 < \lambda \leqslant 1$, *risultando inoltre* $\Theta \neq \Theta^*$ *per* $\lambda \neq 1$ [8]. Difatti la funzione delle due variabili $1/\sqrt{F(u, \lambda)}$, è continua al variare del punto (u, λ) nell'insieme

$$u_1 < u < u_2, \qquad 0 < \lambda \leqslant 1,$$

risultando ivi per la (14)

$$\frac{1}{\sqrt{F(u, \lambda)}} \leqslant \frac{1}{\sqrt{F^*(u)}}.$$

Dalla sommabilità in (u_1, u_2) e dall'indipendenza da λ del secondo membro segue la predetta continuità di $\Theta(\lambda)$; in più si ha $\Theta \leqslant \Theta^*$, sussistendo l'uguaglianza *se e solo se* è $\lambda = 1$.

Sarà facile ora dimostrare la proposizione enunciata in fine alla Sezione 3'1. Supposto di avere appunto, per ogni fissato η compreso tra 0 ed 1, scelto λ in

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[8] In base a ciò risulta anche provato (cfr. Sezione 2, in fine) che effettivamente si presentano, in dipendenza delle condizioni iniziali, entrambi i casi di Θ commensurabile o non con π.

maniera che Θ e Θ^* siano tra loro incommensurabili, introduciamo le variabili

$$\tau(\lambda, t) = \frac{\pi}{\Theta^*}\,\theta(\lambda, t)\,, \qquad \tau^*(t) = \frac{\pi}{\Theta^*}\,\theta^*(t)\,,$$

e, in corrispondenza a σ^*, $\sigma(\lambda)$, consideriamo i movimenti dei punti Q_τ^*, Q_τ le cui coordinate polari per ogni valore di t siano rispettivamente $r^*(t)$, $\tau^*(t)$ e $r(\lambda, t)$, $\tau(\lambda, t)$. Le orbite γ^*, γ di Q_τ^*, Q_τ si svolgono anch'esse nella corona circolare Γ delimitata dalle circonferenze concentriche in O e con i raggi $r_1 = 1/u_1$, $r_2 = 1/u_2$ $(r_1 > r_2)$, toccando alternativamente e periodicamente tali due circonferenze. Ma l'angolo apsidale relativo alla traiettoria γ^* di Q_τ^* è uguale a π e pertanto γ^* è chiusa; invece per la supposta incommensurabilità tra Θ e Θ^*, l'angolo apsidale relativo a γ è certamente non commensurabile con π. Quindi γ non è chiusa, ma essa si avvolge infinite volte attorno ad O riempiendo Γ, nel senso specificato alla Sezione **2**˚2.

Sia ora I un cerchio contenuto in Γ e tale che ad esso non appartenga alcun punto di γ^* ed indichiamo con δ la distanza, certamente positiva, di I da γ^*. Le proprietà di γ assicurano che Q_τ transiti, in corrispondenza di infiniti valori di t, per punti di I; ma allora risulterà

$$|Q_\tau Q_\tau^*| \geqslant \delta\,,$$

donde, tenendo conto che

$$|Q_\tau Q_\tau^*|^2 = r^2 + r^{*2} - 2rr^* \cos(\tau - \tau^*) = r^2 + r^{*2} - 2rr^* \cos\frac{\pi}{\Theta^*}(\theta - \theta^*) =$$

$$= (r - r^*)^2 + 4rr^* \sin^2\frac{\pi}{2\Theta^*}(\theta - \theta^*)\,,$$

si ha

$$(r - r^*)^2 + 4rr^* \sin^2\frac{\pi}{2\Theta^*}(\theta - \theta^*) \geqslant \delta^2\,,$$

e quindi a maggior ragione

$$(r - r^*)^2 \geqslant \delta^2 - 4r_1^2 \sin^2\frac{\pi}{2\Theta^*}(\theta - \theta^*)\,. \tag{16}$$

Ma per essere $\delta > 0$ si potranno certamente determinare i numeri ε_1, ε_2 positivi ed indipendenti da σ tali che

$$\varepsilon_1^2 = \delta^2 - 4r_1^2 \sin^2\frac{\pi\varepsilon_2}{2\Theta^*}\,, \qquad \varepsilon_2 < \Theta^*\,.$$

Onde, se per i predetti valori di t è $|\theta - \theta^*| < \varepsilon_2$, risulta, in virtù della (16) $|r - r^*| > \varepsilon_1$.

Ciò appunto volevasi dimostrare.

3'3. – Prendiamo ora in considerazione i casi dianzi esclusi.

Quando $a = 0$, l'instabilità di σ^* risulta evidente; difatti è ovvia, dopo quanto si è detto, l'esistenza di moti σ di Q, arbitrariamente prossimi nell'istante iniziale a σ^*, e tali che a sia diverso da zero e che u oscilli tra due radici semplici u_1, u_2 di $F(u)$ con un periodo incommensurabile con 2π. La predetta instabilità è allora conseguenza immediata della già richiamata proprietà della traiettoria di *riempire* la corona circolare delimitata dalle circonferenze di raggi $1/u_1$, $1/u_2$.

Supponiamo ora, infine, che σ^* sia un moto circolare (uniforme) di Q, ovvero che i valori iniziali r_0^*, $\dot{r}_0^*$, $\dot{\theta}_0^*$, $\dot{z}_0^*$ rendano soddisfatte le (7).

Anche in questo caso si ha instabilità e la dimostrazione si consegue confrontando il moto in esame con la classe di movimenti che rendono soddisfatte inizialmente le (7) e le seguenti altre

$$\theta_0 = \theta_0^*, \qquad r_0 = r_0^*, \qquad |\dot{\theta}_0 - \dot{\theta}_0^*| = \varepsilon\,, \tag{17}$$

con ε numero positivo e arbitrariamente piccolo. Si tratta quindi di moti circolari uniformi sulla circonferenza di raggio r_0^* e prossimi quanto si vuole a σ^*, pur di scegliere ε convenientemente piccolo.

Dalle identità

$$\dot{\theta} = \dot{\theta}_0\,, \qquad \dot{\theta}^* = \dot{\theta}_0^*\,,$$

nonchè dalle (17_1), (17_3) si ricava

$$|\theta - \theta^*| = \varepsilon t\,,$$

la quale mostra che comunque piccolo si assuma ε, basta attendere un tempo T sufficientemente grande, perchè per $t > T$ risulti $|\theta - \theta^*|$ superiore ad ogni limite prefissato ad arbitrio.

3'4. – Diremo che per σ^* vi è stabilità *ridotta* all'orbita l^*, quando in ogni moto σ inizialmente abbastanza prossimo a σ^*, la distanza di Q da l^* si mantiene arbitrariamente piccola.

Ciò premesso, proveremo che questo tipo di stabilità effettivamente si presenta quando σ^* è un moto circolare; anzi dimostreremo la proposizione un po' più generale che se per un moto σ, caratterizzato con i valori iniziali r_0, θ_0, $\dot{r}_0$, $\dot{\theta}_0$, $\dot{z}_0$, risultano convenientemente piccole le quantità

$$|r_0 - r_0^*|, \qquad |\dot{r}_0|, \qquad |\dot{\theta}_0 - \dot{\theta}_0^*|, \qquad |\dot{z}_0 - \dot{z}_0^*|,$$

le altre

$$|u(\theta) - u_0^*|, \qquad \left|\frac{\mathrm{d}u}{\mathrm{d}\theta}\right|,$$

si conservano inferiori ad un limite prefissato ad arbitrio, per ogni θ.

Cominciamo con l'osservare che per ogni fissato valore iniziale θ_0 di θ, le (3) stabiliscono una corrispondenza biunivoca tra i due sistemi di costanti

$$r_0, \dot{r}_0, \dot{\theta}_0, \dot{z}_0; \qquad u_0, \left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)_{\theta=\theta_0}, \beta, \gamma,$$

essendo inoltre continue le funzioni che esprimono le une mediante le altre; pertanto nella nostra questione potremo sostituire la condizione su espressa, relativa ai dati iniziali, con quella che siano piccole le quantità

$$|u_0 - u_0^*|, \qquad \left|\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)_{\theta=\theta_0}\right|, \qquad |\beta - \beta^*|, \qquad |\gamma - \gamma^*|.$$

Posto ora

$$W(\beta, \gamma, u) = u^2 + (\beta \log u + \gamma)^2,$$

la (5) si scrive

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + W = \alpha^2, \tag{5'}$$

e, stante la forma di tale equazione ed il fatto che $W(\beta^*, \gamma^*, u)$ ha per $u = u_0^*$ un minimo effettivo [9], già l'ordinario criterio di Dirichlet assicura la stabilità predetta, subordinatamente però alle *sole* perturbazioni di σ^* verificanti le condizioni $\beta = \beta^*$, $\gamma = \gamma^*$. Tale limitazione sul carattere della stabilità non è essenziale, così come ora dimostreremo, con l'ausilio di un teorema di LIAPOUNOFF [10].

Posto precisamente

$$x_1 = u - u_0^*, \qquad x_2 = \frac{\mathrm{d}u}{\mathrm{d}\theta}, \qquad x_3 = \beta - \beta^*, \qquad x_4 = \gamma - \gamma^*,$$

$$U(x_1, x_3, x_4) = W(\beta^* + x_3, \gamma^* + x_4, u_0^* + x_1),$$

[9] Ricordiamo (Sezione 2.2) che, in corrispondenza ad un'orbita circolare di raggio r_0 è $1/r_0$ radice multipla di $F(u)$ e che inoltre abbiamo mostrato, a proposito dell'impossibilità di verificarsi del caso b), che se per $u = \bar{u}$ risulta $F'(\bar{u}) = 0$, deve essere necessariamente $F''(\bar{u}) < 0$. La nostra affermazione è allora conseguenza del fatto che W differisce da $-F$ solo per la costante additiva α^2.

[10] A. LIAPOUNOFF: *Problème général de la stabilité du mouvement* (Princeton, 1949), cap. I, n. 16.

la (5′) può sostituirsi con l'equivalente sistema

$$(5'') \qquad x_2^2 + U(x_1, x_3, x_4) = \alpha^2, \qquad \frac{dx_1}{d\theta} = \dot{x}_2, \qquad \frac{dx_3}{d\theta} = 0, \qquad \frac{dx_4}{d\theta} = 0\,,$$

e la nostra questione viene con ciò a tradursi in quella di studiare la stabilità della soluzione

$$x_1 = x_2 = x_3 = x_4 = 0\,,$$

di quest'ultimo.

Consideriamo ora x_1, x_2, x_3, x_4 come coordinate cartesiane ortogonali in uno spazio S_4 (euclideo) a quattro dimensioni e la funzione

$$H = x_2^2 + U,$$

del generico punto R di tale spazio.

Il valore H^* di tale funzione nell'origine Ω delle coordinate è per essa un minimo effettivo subordinatamente alle condizioni $x_3 = x_4 = 0$. Pertanto, indicato con δ un numero positivo convenientemente piccolo, sarà

$$(18) \qquad H > H^*\,,$$

per ogni punto R del piano coordinato di equazioni $x_3 = x_4 = 0$, distinto da Ω e non esterno alla ipersfera Σ_δ di centro Ω e raggio δ.

Notiamo ora che la funzione

$$V = (H - H^*)^2 + x_3^2 + x_4^2\,,$$

è nulla nell'origine mentre che i valori che essa assume in ogni altro punto, non esterno alla ipersfera Σ_μ di centro Ω ed avente un raggio μ qualsiasi minore di δ sono tutti positivi. Difatti per essere V nulla in uno di questi punti dovrebbero essere ivi simultaneamente nulle le quantità positive

$$(H - H^*)^2,\ x_3^2,\ x_4^2;$$

ma ciò è assurdo poichè ove risulti $x_3 = x_4 = 0$, è valida la (18) e quindi non può annullarsi $H - H^*$.

Cioè la V è una *funzione definita positiva* [11] e poichè inoltre in corrispondenza ad ogni soluzione delle (5″) essa si conserva costante al variare di θ, sono verificate tutte le ipotesi del succitato teorema di LIAPOUNOFF, col quale rimane appunto assicurata la stabilità della soluzione in esame.

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[11] Cfr. loc. cit. in [10], n. 15.

3'5. – Supponiamo ora che σ^* sia un movimento in cui u oscilli tra due radici semplici di $F(u)$.

Gli stessi argomenti addotti per dimostrarne l'instabilità (Sezione 3'2), sono ovviamente sufficienti a provare che se più particolarmente l'orbita l^* di σ^* è chiusa, per esso non vi è neppure la stabilità *ridotta* a l^*.

Proponendoci ora di dimostrare la proposizione II enunciata [12] alla Sezione 1, premettiamo l'osservazione che se σ è un moto in cui u oscilla tra due radici semplici u_1, u_2 $(u_1 < u_2)$ di $F(u)$, risulta, in ogni punto esterno all'intervallo (u_1, u_2), $F(u) < 0$. Ciò poi comporta che *se* $F(u)$ *è nulla per i valori* $\overline{u}_1$, $\overline{u}_2$ $(\overline{u}_1 < u_2)$, *non potrà essere che* $\overline{u}_1 = u_1$, $\overline{u}_2 = u_2$.

Difatti la $F(u)$ nulla negli estremi u_1, u_2 e positiva nell'intervallo aperto (u_1, u_2), può scriversi

$$F(u) = (u - u_1)(u_2 - u)F_1(u) ,$$

ove $F_1(u)$ denota una funzione positiva in tutto l'intervallo (u_1, u_2), *estremi inclusi* (per il fatto che u_1 ed u_2 sono radici semplici di $F(u)$). Da ciò segue che

$$\left(\frac{\mathrm{d}F}{\mathrm{d}u}\right)_{u=u_1} > 0 , \qquad \left(\frac{\mathrm{d}F}{\mathrm{d}u}\right)_{u=u_2} < 0 , \tag{19}$$

e che pertanto la $F(u)$ in un intorno sinistro di u_1, ed analogamente in un intorno destro di u_2 (i due intorni privati naturalmente dei punti u_1, u_2), prende valori tutti negativi. Per dimostrare la predetta osservazione basterà allora provare che esternamente all'intervallo (u_1, u_2) non potrà mai essere

$$\frac{\mathrm{d}F}{\mathrm{d}u} = 0 .$$

Ora in virtù della (4) si ha

$$\frac{\mathrm{d}F}{\mathrm{d}u} = -\frac{2}{u}\left[u^2 + \beta(\beta \log u + \gamma)\right] ,$$

e quindi per le (19)

$$u_1^2 + \beta(\beta \log u_1 + \gamma) < 0 , \qquad u_2^2 + \beta(\beta \log u_2 + \gamma) > 0 .$$

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[12] In detto enunciato è contenuta implicitamente l'ammissione che in tutti i moti σ inizialmente abbastanza prossimi a σ^*, così come in σ^* medesimo, u oscilli tra due radici semplici di $F(u)$. Omettiamo di fornire la relativa dimostrazione che si presenta pressocchè immediata.

Pertanto la funzione crescente

$$u^2 + \beta(\beta \log u + \gamma) ,$$

sarà negativa per $u < u_1$ e positiva per $u > u_2$ cioè, appunto, mai nulla esternamente all'intervallo (u_1, u_2).

Ciò premesso, indichiamo con $F(r_0, \dot{r}_0, \dot{\theta}_0, \dot{z}_0, u)$ la funzione $F(u)$ medesima considerata anche nella sua dipendenza dai valori iniziali. Poichè i valori estremi u_1^*, u_2^* tra cui oscilla u in σ^* sono *radici semplici* di $F(r_0^*, \dot{r}_0^*, \dot{\theta}_0^*, \dot{z}_0^*, u)$, l'equazione

$$F(r_0, \dot{r}_0, \dot{\theta}_0, \dot{z}_0, u) = 0 ,$$

definisce due funzioni implicite $\overline{u}_1$, $\overline{u}_2$ di $r_0, \dot{r}_0, \dot{\theta}_0, \dot{z}_0$ tali che

$$\overline{u}_1(r_0^*, \dot{r}_0^*, \dot{\theta}_0^*, \dot{z}_0^*) = u_1^* , \qquad \overline{u}_2(r_0^*, \dot{r}_0^*, \dot{\theta}_0^*, \dot{z}_0^*) = u_2^* ,$$

quando si abbia

$$(20) \qquad |r_0 - r_0^*| < \varepsilon_1 , \quad |\dot{r}_0 - \dot{r}_0^*| < \varepsilon_2 , \quad |\dot{\theta}_0 - \dot{\theta}_0^*| < \varepsilon_3 , \quad |\dot{z}_0 - \dot{z}_0^*| < \varepsilon_4 ,$$

ove i numeri ε_i sono positivi ed abbastanza piccoli.

Consideriamo infine un moto σ tanto prossimo inizialmente a σ^* che [13], innanzi tutto, u oscilli tra due radici semplici u_1, u_2 di $F(r_0, \dot{r}_0, \dot{\theta}_0, \dot{z}_0, u)$ e, inoltre, siano verificate le (20).

Ma, allora, per la premessa osservazione sarà

$$u_1 = \overline{u}_1 , \qquad u_2 = \overline{u}_2 ,$$

dopo di che la proposizione II è conseguenza della continuità, nelle nostre condizioni certamente assicurata, delle funzioni implicite $\overline{u}_1, \overline{u}_2$.

* * *

Si ringraziano, per utili discussioni, i Proff. C. Tolotti e E. R. Caianiello.

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[13] Cfr. nota [12].

SUMMARY

A complete and rigorous analysis is presented of the stability of motions of charged particles in a Biot-Savart field. This situation, which was already discussed partially by Caianiello and Gatti because of its possible interest, as a test case, in problems regarding the design of accelerators or the confinement of accelerated plasma ions, is studied here in detail; it is shown, in particular, that the customary approximate methods for the study of stability may easily lead to wrong conclusions.

Analysis of Some Nuclear Interactions of Energies $(10^{12} \div 10^{14})$ eV/Nucleon.

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(ricevuto il 16 Agosto 1960)

Summary. — Five nuclear interactions produced by nuclei with values of charge $Z=1$, 2, 6, 7 and 14 and with energies between 10^{12} and 10^{14} eV/nucleon are described. The angular distributions and the momentum distributions of the secondary particles produced in these jets have been studied. Assuming transverse momentum to be constant, the momentum spectrum in the C.M. system in the case of each of the three jets with energies between $(10^{13} \div 10^{14})$ eV/nucleon is found to be of the form $P^{-2}\,\mathrm{d}P$. Twenty nuclear interactions produced by secondary particles arising in two of the jets which involve energies of about 10^{14} eV/nucleon have also been analyzed. The ratio of the number of secondary interactions produced by neutral to that produced by charged particles is found to be 0.23 ± 0.15. The energies of the particles responsible for 16 of the secondary interactions have been determined. The transverse momentum distribution of the secondary particles, produced in nucleon-nucleus and nucleus-nucleus collisions at $\sim 10^{14}$ eV/nucleon and emitted at small angles (*i.e.* belonging to the inner cone), is found to be consistent, in the cases studied here, with that observed in nucleon-nucleon collisions at this energy. The observations indicate that nucleon-nucleus and nucleus-nucleus collisions at energies of about $(10^{13} \div 10^{14})$ eV/nucleon are, in general, similar to nucleon-nucleon collisions.

1. – Introduction.

During the last few years a large number of high energy nuclear disintegrations, generally referred to as *jets*, produced by cosmic ray particles with energies $\geqslant 5 \cdot 10^{10}$ eV/nucleon have been observed in nuclear emulsions. The

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number of useful jets at *ultra high energies*, *i.e.*, primary energies $\geqslant 10^{13}$ eV/nucleon is still very meagre ([1-19]) and most of the observed ones are found to be produced by protons and α-particles and only very few by heavier nuclei ([1,4]). A comparison of the characteristics of jets produced by nucleons with those produced by heavier nuclei is of interest in studying differences, if any, in the mechanism of nucleon-nucleon, nucleon-nucleus and nucleus-nucleus collisions at ultra high energies. If the features of nucleon-nucleus and nucleus-nucleus collisions are similar to those of nucleon-nucleon collisions then, in general, because of the higher meson multiplicities resulting from the former, they will offer statistically better samples for analysis. Also, a knowledge concerning the highest energies of the singly and multiply charged primary particles which produce such interactions is of importance for understanding the origin and the acceleration mechanisms of cosmic radiation.

We describe in this paper five jets observed in nuclear emulsions. Out of these, three are produced by primary particles with estimated energies

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([1]) D. Lal, Yash Pal, B. Peters and M. S. Swami: *Proc. Ind. Acad. Sci.*, A **36**, 75 (1952).

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([3]) R. G. Glasser, D. M. Haskin, M. Schein and J. J. Lord: *Phys. Rev.*, **99**, 1555 (1955).

([4]) J. E. Naugle and P. S. Freier: *Phys. Rev.*, **104**, 804 (1956).

([5]) F. D. Hanni, C. Lang, M. Trucher, H. Winzeler and E. Lohrmann: *Nuovo Cimento*, **4**, 1473 (1956).

([6]) P. H. Fowler: *VII Rochester Conference on High Energy Nuclear Physics* (1957); C. F. Powell, P. H. Fowler and D. H. Perkins: *The Study of Elementary Particles by the Photographic Method* (London, 1959).

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([8]) I. M. Gramenitskii, G. B. Zhdanov, E. A. Zamchalova and M. N. Scherbakova: *Žurn. Èksp. Teor. Fiz.*, **32**, 936 (1957).

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([10]) A. P. Zhdanov, I. B. Berkovich, K. I. Ermakova, F. G. Lepekhin, N. V. Skirda and Z. S. Khokhlova: *Dokl. Acad. Nauk, SSSR.*, **115**, 1093 (1957).

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([12]) B. Edwards, J. Losty, D. H. Perkins, K. Pinkau and J. Reynolds: *Phil. Mag.*, **3**, 237 (1958).

([13]) S. Hasegawa: *Nuovo Cimento*, **14**, 909 (1959).

([14]) F. A. Brisbout and C. B. A. McCusker: *Nuovo Cimento*, **11**, 484 (1959).

([15]) M. W. Teucher, E. Lohrmann, D. M. Haskin and M. Schein: *Phys. Rev., Lett.*, **2**, 313 (1959).

([16]) K. Nishikawa: *Journ. Phys. Soc. (Japan)*, **14**, 880 (1959).

([17]) G. Bozoki and E. Gombosi: *Nucl. Phys.*, **9**, 400 (1959).

([18]) J. G. McEwen: *Phys. Rev.*, **115**, 1712 (1959).

([19]) G. Bozoki, G. Pomokos, E. Fenyves, A. Frenkel, E. Gombosi, D. Behel, K. Lanius and H. W. Meier: *Nuovo Cimento*, **13**, 662 (1959).

$\sim 10^{14}$ eV/nucleon and in the other two the primary particles have estimated energies $\sim 10^{13}$ and $\sim 10^{12}$ eV/nucleon. The multiplicities in all of these jets are rather high. Attempts have been made to study the momentum distribution and the angular distribution of the secondary particles. We have also analysed 20 interactions produced by secondary particles arising in two jets with primary energies $\sim 10^{14}$ eV/nucleon. We have studied the transverse momentum distribution of the particles responsible for these secondary interactions and also the angular distributions of the tertiary particles.

Ultra high energy interactions, and in particular those produced by heavy nuclei, are quite rare. It is therefore considered desirable to publish details of events in this category as are observed. We have, in our description of these events in this paper, included the raw data obtained from the measurements, so that other workers in the field may use this information for their own analysis or interpretation.

2. – Primary interactions.

The five high energy interactions were obtained in a stack consisting of sixty-four, 8 in. × 6 in. × 600 μm, G-5 stripped emulsions exposed on a flight from Texas, U.S.A., at an atmospheric depth of 5.3 g cm^{-2} for 6 hours and 50 minutes. We shall, hereafter, designate (*) these jets as *J*2, *J*3, *J*4, *J*5 and *J*6. To describe these jets we have followed the nomenclature introduced by the Bristol group ([20,21]). The three jets (*J*2, *J*3 and *J*6), produced by heavy nuclei, were obtained in the course of an experiment involving the systematic following of tracks produced by heavy primary cosmic-ray particles of charge $\geqslant 6$. In this systematic following if there were energetic events involving low multiplicities ($n_s < 20$) they would not have been included for analysis. In the same stack two more jets were observed, one produced by a singly charged particle (*J*4) and the other by a doubly charged one (*J*5).

2·1. *Description of the events.* – The general features of the jets are listed in Table I.

The charges of the primary particles responsible for the jets *J*2, *J*3 and *J*6 were established by measurements of δ-ray densities. All the three heavy nuclei suffered complete disintegration *i.e.*, no energetic fragment with $Z > 1$ was observed among the secondary particles. It should be mentioned that jets produced by heavy nuclei have been observed in other laboratories ([4])

(*) The magnesium jet ([1]) reported earlier from this laboratory is designated as *J*1-jet.

([20]) R. H. Brown, U. Camerini, P. H. Fowler, H. Heitler, D. T. King and C. F. Powell: *Phil. Mag.*, **40**, 862 (1949).

([21]) U. Camerini, J. H. Davies, P. H. Fowler, C. Franzinetti, W. O. Lock, H. Muirhead, D. H. Perkins and G. Yekutieli: *Phil. Mag.*, **42**, 1241 (1951).

TABLE I.

Jet	Charge Z of primary particle	Zenith angle	Size	Length per plate (mm)	Total length in stack (cm)	Estimated energy (eV/nucleon)
$J2$	14	32°	$1+1+(\sim 200)$	0.79	1.7	$\sim 5 \cdot 10^{14}$
$J3$	6	43°	$11+7+160$ (*) (182)	1.18	6.8	$8.0 \cdot 10^{13}$
$J4$	1	52°	$6+0+44$ (50)	1.02	5.4	$8.0 \cdot 10^{13}$
$J5$	2	63°	$2+1+29$	0.87	1.4	$1.4 \cdot 10^{13}$
$J6$	7	39°	$0+1+37$	1.83	11.0	$1.2 \cdot 10^{12}$

(*) Three tracks due to shower particles from the $J3$-jet have angles greater than 90°. In all the other events not one such track has been observed in the backward direction in the L-system.

in which the heavy nuclei do not undergo complete fragmentation. In the absence of sufficient data, it is difficult to say to what extent complete fragmentation occurs in the very energetic collisions of heavy primary nuclei with various emulsion nuclei. In Table I, the numbers given inside brackets in column 4 refer to the total numbers of shower particles found at the points of observation. These numbers were then corrected for the contributions due to electrons and positrons arising through the decay of neutral pions (see Section 2·2) and the corrected numbers of shower particles are shown in the same column. For the jets $J5$ and $J6$ the multiplicities were determined close to the interaction points and so the contamination due to the development of the electromagnetic cascade is negligible.

In the case of the $J2$-jet the multiplicity is ~ 200 and the collimation of the shower particles is even greater than that of the $J3$-jet. It was not possible to make accurate angular measurements on individual tracks even at a distance of 1.2 cm from the primary interaction; at this point there is a secondary high energy interaction with about 50 well collimated particles; as a result of this, angular measurements on the secondary particles from $J2$-jet were more or less impossible. We have, therefore, not been able to analyse this event.

One of the black prongs of $J4$-jet is most probably due to a hyperfragment which decays mesonically. The fragment is very steep and has a length $\sim 30\ \mu$m only. It decays into two non-collinear charged particles which respectively give rise to a short black track and a light track which is very steep and eventually leaves the stack.

The jets $J3$, $J4$ and $J5$ are interesting events because of the high energy of the primary particles, the high multiplicity of the shower particles and their suitability for analysis. We have also analysed in the same manner the jet $J6$ which is of relatively low energy.

2·2. *Angular distribution of shower particles and estimation of primary energy.* – Angular measurements on the shower particles in the cores were carried out under a total magnification of 1500. Target patterns were made at the minimum distances, from the points of interaction, where the individual tracks of the shower particles were just resolved enough. In the cases of the jets $J3$, $J4$, $J5$ and $J6$ these distances were 5.82, 4.00, 1.36 and 0.53 mm respectively; 16% of the distance, in each of these cases, is in the air gap between emulsions. Since the target patterns for the jets $J5$ and $J6$ were obtained fairly close to the origins, this being possible because of the relatively low multiplicities and energies, the contamination arising from the development of the electromagnetic cascade should be negligible, as can be shown easily. The jets $J3$ and $J4$ involve much higher energies and larger multiplicities and therefore the target patterns had to be made at 5.82 mm and 4.00 mm respectively from the origins. It is estimated that 22 tracks in the $J3$-jet and 6 tracks in the $J4$-jet at the points where the target patterns were made are due to electrons and positrons arising from π^0-decays. These numbers have been deducted from the observed multiplicities in order to deduce the true multiplicities. For purposes of our analysis we assume that the angular distribution of these electrons and positions is the same as that of the other shower particles.

2·2.1. Estimation of primary energy. – If we assume that interactions can be analyzed in terms of one or more independent nucleon-nucleon collisions, then the Lorentz factor γ_c of the centre of momentum system (C.M. system) is related to the median angle in the L-system, θ_M, as

$$\gamma_c \operatorname{tg} \theta_M = 1 \tag{1}$$

on the assumptions that: i) the ratio of the velocity of the shower particles in the C.M. system to that of the C.M. system is close to unity; and ii) there is forward-backward symmetry of shower particles in the C.M. system. The primary energy in the L-system is then obtained from the relation

$$\gamma_L = 2\,\gamma_c^2 - 1\,. \tag{2}$$

J2-jet. As mentioned earlier in Section 2·1, we have not been able to make angular measurements on individual tracks of the shower particles in the $J2$-jet and so it has not been possible to determine the energy of the primary particle in a manner similar to that employed for the other jets. However, we have made a visual estimate of the opening angle of the inner cone and obtained a value $\sim 5 \cdot 10^{14}$ eV/nucleon (Table II) for the primary energy.

J3 and J4 jets. To obtain the correct value of θ_M, we have plotted, in Fig. 1, the quantity $(F/(1-F))$ as a function of θ—the so-called F-plot; F is the ratio of the number of shower particles with angles less than θ in the L-system to the total number of shower particles ([22]). Figs. 1-*A* and 1-*B* show

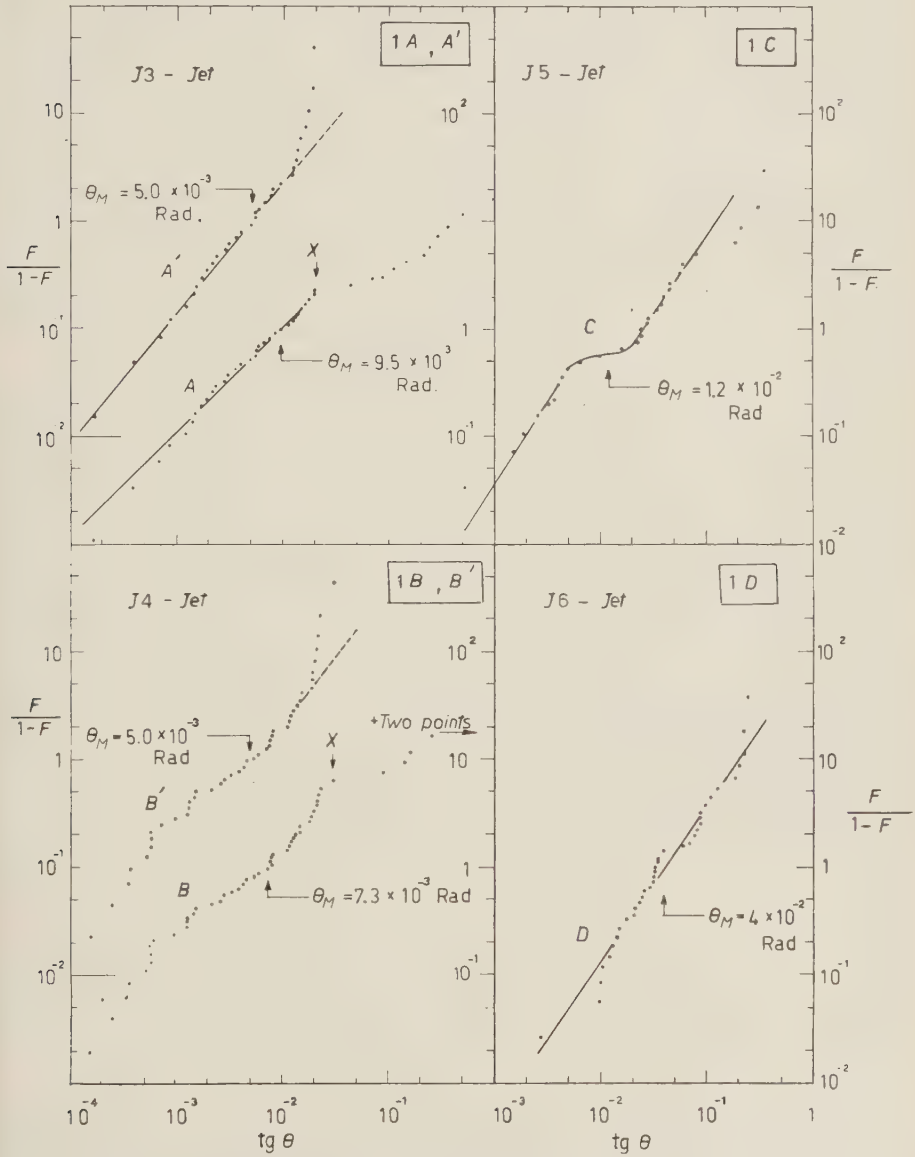

Fig. 1. – Plots of $[F/(1-F)]$ against $\mathrm{tg}\,\theta$ on a log-log scale for the jets *J3*, *J4*, *J5* and *J6*.

([22]) N. M. Duller and W. D. Walker: *Phys. Rev.*, **93**, 215 (1954).

the F-plots for all shower particles associated with jets $J3$ and $J4$. It may be noted that in neither of these there is a clear indication of a « double-cone » structure ([23-25]). These plots run smoothly till the point marked X, at which place there is a break in the continuity and a change of slope after the break. From this it would appear that the particles at angles greater than that determined by the point X do not arise in the primary act but are probably produced in secondary processes, such as collisions of the shower particles with target nucleons outside the main « tunnel ». Henceforth we shall refer to this as the « secondary effect ». Most of the particles arising in such secondary processes are expected to be at large angles. Two further points relevant to this may also be stated. The jet $J3$ has a value of $N_h = 18$, which is suggestive of secondary collisions having occurred. Also, the primary energy estimated below, from the angular distribution of all the shower particles, is inconsistent with the high energies observed for the secondary interactions; this indicates that all of the shower particles, particularly the ones at large angles, have not arisen in the same primary collision.

In Figs. 1-A' and 1-B' are shown the F-plots for the events $J3$ and $J4$ after neglecting the shower particles at large angles which are attributed to the « secondary effect ». The steepness of the plots A' and B' at comparatively large angles is due to the fact that in neglecting all the shower particles beyond X we have probably removed some which arose in the primary process and also there may still be some due to the « secondary effect » at angles smaller than that corresponding to the point X. So far as the *true multiplicity* (n_s) in the main interaction is concerned, the two effects probably cancel each other to some extent. The true curves at large angles for these two jets are very likely determined by the extrapolation of the curves fitted to the points at smaller angles. These are shown as dotted lines in Fig. 1 as A' and B'. It is reasonable, therefore, to assume that the primary energy estimated from the median angle given by the F-plot for all shower particles (plot A or B) is only a lower limit whilst that obtained from the corrected F-plot (A' or B') is the probable value.

J5-jet. It can be seen from Fig. 1-C that the F-plot for $J5$-jet indicates a well resolved « *double-cone* » structure and, therefore, in this case θ_M is defined by the centre of the « plateau ». Here, we can also determine the primary

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([23]) P. Ciok, T. Coghen, J. Gierula, R. Hołynski, A. Jurak, M. Mięsowicz, T. Saniewska, O. Stanisz and J. Pernegr: *Nuovo Cimento*, **8**, 166 (1958); P. Ciok, T. Coghen, J. Gierula, R. Hołynski, A. Jurak, M. Mięsowicz and T. Saniewska: *Nuovo Cimento*, **10**, 741 (1958).

([24]) G. Cocconi: *Phys. Rev.*, **111**, 1699 (1958).

([25]) K. Niu: *Nuovo Cimento*, **10**, 994 (1958).

energy on the basis of the «*two-centre*» *model* ([23,25]). Fig. 2 shows two separate F-plots for the particles assigned to the lumps moving forward and backward in the C.M. system. The slopes of the two graphs are 2.1 and 2.6 and are consistent with isotropic emission from each lump. The two lumps are characterized by median angles $3.3 \cdot 10^{-3}$ and $4.2 \cdot 10^{-2}$ radians which correspond to Lorentz factors, in the L-system, $\gamma_{b1}=298$ and $\gamma_{b2}=23$, giving a value $\gamma_b^*=\frac{1}{2}(\gamma_{b1}/\gamma_{b2})^{\frac{1}{2}}=1.8$ for the Lorentz factor of each lump in the C.M. system. The Lorentz factor for the C.M. system is determined by the relation $\gamma_c = (\gamma_{b1} \cdot \gamma_{b2})^{\frac{1}{2}} = 83$. The primary energy is thus obtained as $1.4 \cdot 10^{13}$ eV/nucleon which is the same as that obtained from the median angle defined by the F-plot shown in Fig. 1-*C*. This agreement indicates that the event *J*5 is a fairly good example of «double-cone» structure.

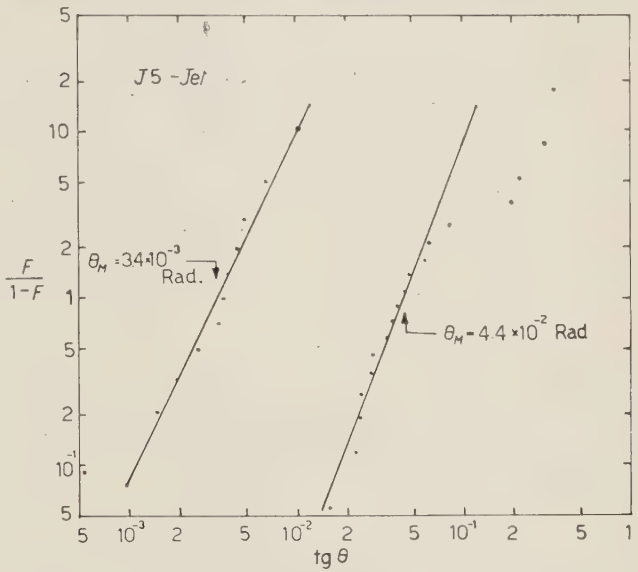

Fig. 2. – Plot of $[F/(1-F)]$ against tg θ in the case of the *J*5-jet; the two sets of points and the lines fitted to them correspond to the two lumps which move forward and backward in the C.M. system.

J6-jet. The F plot for the *J*6-jet is shown in Fig. 1-*D*. There is no clear indication of a «double-cone» structure in this plot. There is no black prong and only one grey prong associated with this jet and there is no discontinuity in the F-plot at large angles. The energy was estimated in a straightforward manner from the median angle.

Primary energies. Energy estimates for the jets *J*2 to *J*6 obtained as described in the preceeding paragraphs are listed in Table II.

For the evaluation of primary energy it was assumed that a nucleus-nucleus collision can be considered to be the superposition of individual nucleon-nucleon collisions. If, however, we treat the collision as between two bodies, then γ_L is given by

$$\gamma_L = 2\gamma_c'^2 \frac{M_t}{M_i}, \tag{3}$$

where γ_c' is the Lorentz factor of the centre of momentum system; M_i and M_t are the masses of the colliding parts of the incident and the target nuclei.

TABLE II.

Jet	Primary energy (eV/nucleon)	
	lower limit	probable value
$J2$	—	$(\sim 5 \cdot 10^{14})$
$J3$	$1.1 \cdot 10^{13}$	$8.0 \cdot 10^{13}$
$J4$	$3.8 \cdot 10^{13}$	$8.0 \cdot 10^{13}$
$J5$	—	$1.4 \cdot 10^{13}$
$J6$	—	$1.2 \cdot 10^{12}$

If we further make the assumption that the angular distribution of the shower particles in this C.M. system is symmetric, backwards and forwards, then γ_c' is related to the median angle θ_M as

$$\gamma_c' \operatorname{tg} \theta_M = 1 \,. \tag{4}$$

This gives for γ_c' the same value as obtained earlier on the assumption of independent nucleon-nucleon collisions. Thus the extent to which we underestimate energy in the L-system by using equation (2), if the assumptions involved are incorrect, depends directly on the value of M_t/M_i. For complete disintegration of the primary nucleus, M_i can be assumed to be nearly equal to the mass of the incident nucleus and M_t is determined by the mass of the tunnel punched in the target nucleus. From geometrical considerations of a central collision between a primary nucleus of medium size (C, N or O) and a silver nucleus, it can be seen that

$$\frac{M_t}{M_i} = \frac{3}{2} \cdot \left[\frac{A(\mathrm{Ag})}{A(\mathrm{N})}\right]^{\frac{1}{3}} \simeq 3.0 \,. \tag{5}$$

Thus for a primary nucleus of medium size, the energy in the L-system can be *underestimated* in 2·2.1 by a maximum factor ~ 3.

2·2.2. Angular distribution of shower particles. – Fig. 3 shows the integral angular distributions in the L-system for the four jets $J3$, $J4$, $J5$ and $J6$. It can be seen that in no case does a single straight line fit all of the points; at least two straight lines with different slopes are needed in each event. The straight lines are of the form $N(<\theta) \propto \theta^l$, where $l = 1.1$, 0.9, 1.1 and 1.3 for the shower particles in the inner cone and 0.5, 0.4, 0.4 and 0.3

for those in the outer cones of the jets $J3$, $J4$, $J5$ and $J6$ respectively. Thus, for the events obtained by us, the inner cone angular distributions can be represented as $N(<\theta) \propto \theta$ in the primary energy range of $(10^{13} \div 10^{14})$ eV/nucleon.

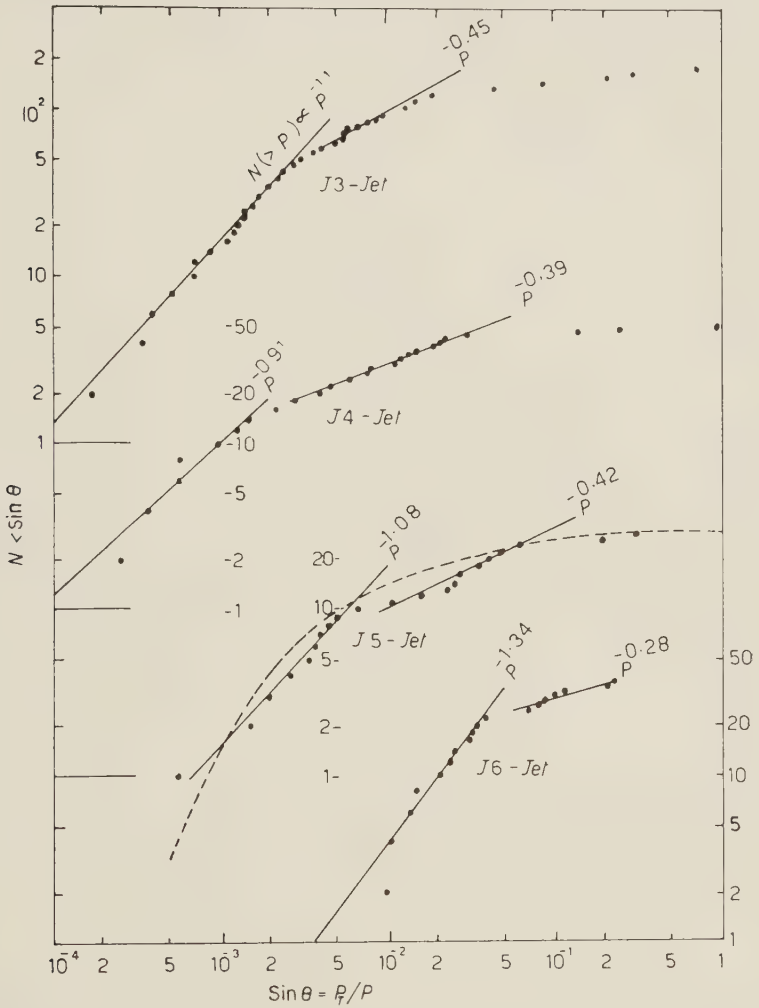

Fig. 3. – The integral angular distributions, in the L-system, for charged shower particles from the jets $J3$, $J4$, $J5$ and $J6$. The broken curve in the case of the $J5$-jet is drawn on the basis of the « two-centre » model.

HASEGAWA *et al.* [26] had arrived at preliminary conclusions of the same type in a study of the spatial distribution of μ-mesons in extensive air showers

[26] S. HASEGAWA, J. NISHIMURA and Y. NISHIMURA: *Nuovo Cimento*, **6**, 979 (1957).

recorded by the Cornell array and by analysing a number of individual jets for which the details were available in published form then.

In Fig. 4, we have plotted the angular distribution of shower particles in the C.M. system for each of the four jets. In this figure, (*a*) and (*b*) show, for the *J*3 and *J*4 jets, the distributions which correspond to the lower limits of

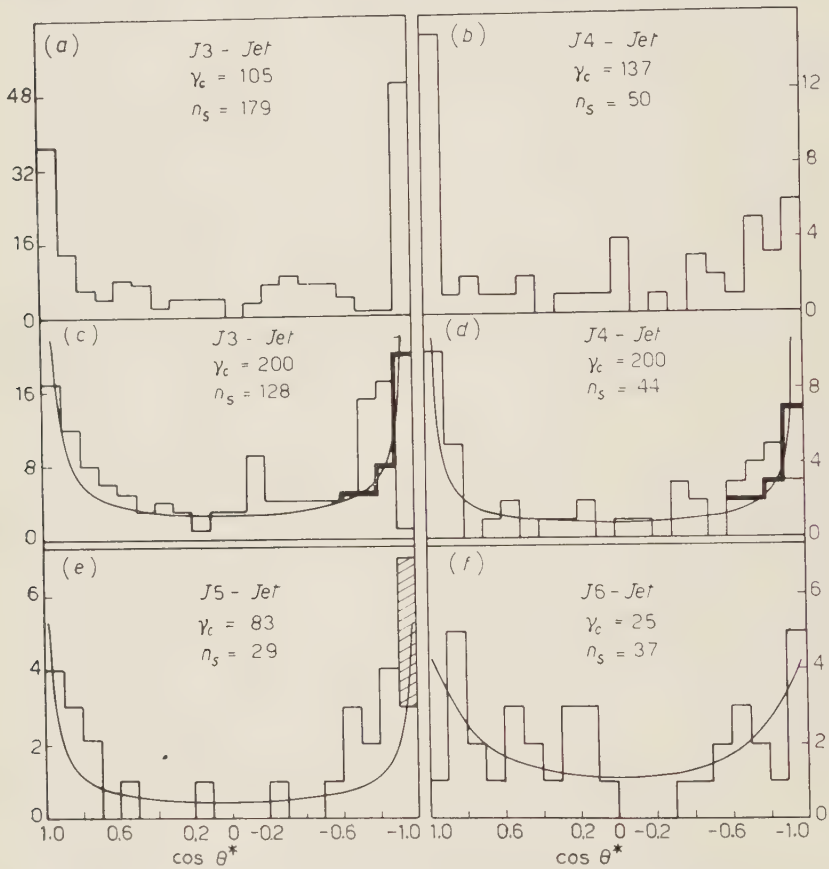

Fig. 4. – The histograms give the angular distributions, in the C.M. system, for shower particles from the *J*3, *J*4, *J*5 and *J*6 jets. The parts indicated by heavy lines and the shaded portion are discussed in the text. The solid curves are drawn on the basis of Landau's theory.

the energy (column 2 in Table II) as estimated using all the shower particles n_0; for the same jets, (*c*) and (*d*) show the distributions which correspond to the probable values of energy (column 3 in Table II). The heavy lines in parts of Fig. 4 (*c*) and (*d*) are drawn on the assumption that the dotted curves in Fig. 1, (*A'*) and (*B'*) represent the correct trend of the *F*-plots at large angles.

As can be seen, this makes the distributions symmetrical with respect to the forward and backward directions in the C.M. system, thereby justifying our procedure in extrapolating the curves in Figs. 1-*A'* and 1-*B'*. The *J*5 and *J*6 jets, of course, need no correction as mentioned earlier and so we have drawn only one plot for each of them. However, there is an indication that the four tracks at large angles in Fig. 1-*C* and shown shaded in Fig. 4(*e*) are due to the « secondary effect », but in view of their small number we have not rejected them. The curves in Fig. 4*c*, 4*d*, 4*e* and 4*f* are drawn on the basis of Landau's theory ([27,28]). The experimental and the theoretical distributions are in good agreement with each other. It is evident that the distributions are anisotropic, at least for the *J*3, *J*4 and *J*5 jets. In this connection we would like to mention that the average slopes of the *F*-plots are found to be 1.0, 0.9, 1.1 and 1.5 for the *J*3, *J*4, *J*5 and *J*6 jets respectively. (In the *F*-plot a slope of 2 indicates an isotropic distribution in the C.M. system. In Heisenberg's theory the distribution is a straight line of slope 1 and in Landau's theory it has an average slope of 1 at 10^{13} eV/nucleon).

c) *Inelasticity*: The inelasticity is generally defined as the fraction of the total available kinetic energy, in the C.M. system, which is carried away by newly created particles. In the case of nucleus-nucleus collisions, or nucleon-nucleus collisions involving further interactions within the same nucleus, it is difficult to obtain a meaningful value for the inelasticity since one does not know how many nucleon pairs have participated in the interaction. The *J*5-jet, however, is produced by an α-particle, and so it is reasonable to assume that either one or at the most two nucleon-pairs have contributed to meson production; the multiplicity in this event is 29 which is not very high. In the « *two-centre* » or « *fireball* » *model*, according to which the particles are emitted isotropically, with an average momentum $\langle P^{**} \rangle$, from two lumps moving forward and backward in the C.M. system—each lump having a Lorentz factor γ_b^*—a measure of the inelasticity is given ([24]) by

$$K' = \frac{1.5 n_s \gamma_b^* \langle P^{**} \rangle}{2\gamma_c} = \frac{3 n_s \gamma_b^* \langle P_T \rangle}{\pi \gamma_c} . \tag{6}$$

Since P_T is known to be nearly constant, a measure of the inelasticity may also be calculated from the formula

$$K'' = \frac{1.5 P_T \sum \operatorname{cosec} \theta^*}{2\gamma_c} , \tag{7}$$

([27]) L. D. Landau: *Izv. Akad. Nauk, SSSR*, **17**, 51 (1953).
([28]) S. Z. Belenkij and L. D. Landau: *Suppl. Nuovo Cimento*, **3**, 15 (1956).

as given by EDWARDS *et al.* ([12]). In both of these relations (6) and (7) it is assumed that all the newly created particles are pions.

Assuming that $\langle P_\pi \rangle = 0.5$ GeV/c, we obtain, for the $J5$ jet, $K' = 0.26/n_i$ and $K'' = 0.28/n_i$, where n_i is the number of nucleon-pairs which have participated in the interaction. Then $K' = 0.26$ and $K'' = 0.28$ if the event represent a nucleon-nucleon collision.

3. – Secondary interactions.

A strip measuring 300 μm in width on either side of the axis of jets $J3$ and $J4$ was scanned for secondary interactions by the method of area scanning, under a total magnification of 1 500; in all 20 secondary interactions were obtained

TABLE III.

Secondary interaction	Size	Angle of the primary (*) (rad)	Primary energy (eV)	
			Castagnoli-method	F-plot
$J3$-1	1+(∼4)?	∼0		(∼ $6.9 \cdot 10^{13}$) (**)
$J3$-2	4+10p	$1.6 \cdot 10^{-4}$	—	(∼ $1.9 \cdot 10^{12}$) (**)
$J3$-3	2+10p	$8.4 \cdot 10^{-4}$	$7.3 \cdot 10^{11}$	$1.2 \cdot 10^{12}$
$J3$-4	4+(∼4)?	$2.5 \cdot 10^{-3}$	—	—
$J3$-5	7+11p	$3.1 \cdot 10^{-3}$	$3.7 \cdot 10^{10}$	$2.4 \cdot 10^{10}$
$J3$-6	4+10p	$3.3 \cdot 10^{-3}$	$6.0 \cdot 10^{11}$	$9.3 \cdot 10^{11}$
$J3$-7	1+ 1p	$5.1 \cdot 10^{-3}$		—
$J3$-8	2+ 2p	$5.6 \cdot 10^{-3}$		—
$J3$-9	3+ 2p	$6.2 \cdot 10^{-3}$		—
$J3$-10	10+20p	$1.1 \cdot 10^{-2}$	$4.7 \cdot 10^{10}$	$2.9 \cdot 10^{10}$
$J3$-11	3+ 4p	$2.6 \cdot 10^{-2}$	$1.6 \cdot 10^{10}$	$2.1 \cdot 10^{10}$
$J3$-12	15+10p	$3.9 \cdot 10^{-2}$	$7.2 \cdot 10^{9}$	$4.7 \cdot 10^{9}$
$J3$-13	2+ 5n	$3.6 \cdot 10^{-3}$	$2.7 \cdot 10^{11}$	$1.8 \cdot 10^{11}$
$J3$-14	3+ 9n	$4.7 \cdot 10^{-3}$	$2.2 \cdot 10^{12}$	$8.1 \cdot 10^{11}$
$J3$-15	19+18n	$1.1 \cdot 10^{-2}$	$5.9 \cdot 10^{10}$	$6.6 \cdot 10^{10}$
$J4$-1	9+13p	$9.2 \cdot 10^{-4}$	$1.3 \cdot 10^{11}$	$5.8 \cdot 10^{10}$
$J4$-2	6+10p	$1.6 \cdot 10^{-3}$	$3.9 \cdot 10^{11}$	$2.9 \cdot 10^{11}$
$J4$-3	3+ 6p	$4.4 \cdot 10^{-3}$	$1.2 \cdot 10^{10}$	$5.6 \cdot 10^{9}$
$J4$-4	0+ 8p	$9.5 \cdot 10^{-3}$	$2.8 \cdot 10^{11}$	$3.8 \cdot 10^{11}$
$J4$-5	1+14n	$5.3 \cdot 10^{-3}$	$4.5 \cdot 10^{10}$	$2.1 \cdot 10^{10}$

(*) This is the angle in the L.S. between primary of the secondary interaction and axis of primary jet.

(**) This is obtained from a rough estimate of the opening-angle.

in all, 15 for the $J3$-jet and 5 for the $J4$-jet. Taking into account the presence of electrons and positrons (which has been discussed in Section 2·2), we obtain for λ, the interaction mean-free-path of the charged shower particles in the cores of events $J3$ and $J4$, values of 40 and 35 cm respectively. Combining the two we get $\lambda = (39 \pm 10)$ cm as compared to 35 cm obtained for protons and pions in the range of a few GeV. The general characteristics of these secondary interactions are summarised in Table III.

The observed value of the ratio of the number of interactions in the core produced by neutral to the number produced by charged particles is $Q = 4/14 = 0.29 \pm 0.16$. In arriving at this we have neglected two events J3-1 and J3-4 in each of which it is not possible to say whether the primary particle is neutral or charged, since these events are located in the dense shower core. In order to estimate the proportion of strange particles and nucleon-antinucleon pairs among the secondary particles, this ratio must be corrected for the contribution of nucleons which result from the breakup of the incident and target nuclei. For determining this, it is reasonable to assume that all nucleons which belonged to the incident nucleus form a part of the shower core, whilst those of the target nucleus do not. For the jets $J3$ and $J4$ taken together, we expect, in the area surveyed, about one proton- and one neutron-initiated secondary interaction. Accordingly, we obtain $N_{\chi^0}/(N_{\chi^\pm} + N_{\pi^\pm}) = 3/13 = 0.23 \pm 0.15$, where N_{χ^0} and $N_{\chi^\pm}$ refer to neutral and charged secondary particles other than pions. If we also include the observations [1] on event $J1$, we obtain $Q = 6/22 = 0.27 \pm 0.12$ and $N_{\chi^0}/(N_{\chi^\pm} + N_{\pi^\pm}) = 4/20 = 0.20 \pm 0.11$. PERKINS [29], in a « world survey », has summarized the available data till 1958, and if we also include in this the data [18] published later, we find that $Q = 24/114 = 0.21 \pm 0.05$. We assume now that these 138 secondary interactions are due only to the particles in the forward cones of the various jets; also that, in general, the number of nucleons present before the collision, and which come into the forward cone of a jet is equal to the total number of nucleons which constituted the incident nucleus (*i.e.* 1 for a proton-initiated jet and 4 for an α-particle-initiated jet). If we assume further that the charge exchange probability is 0.5, then of these 138 interactions, 7 can be attributed to the « initial » neutrons and another 7 to « initial » protons. Accordingly, the ratio $N_{\chi^0}/(N_{\chi^\pm} + N_{\pi^\pm}) = 17/107 = 0.16 \pm 0.04$. The results obtained in this investigation are consistent with those resulting from the « world survey ».

The energies of the particles responsible for the secondary interactions were determined by the F-plot method as well as by Castagnoli's statistical

[29] D. H. PERKINS: *Progress in Elementary Particles and Cosmic Rays*, vol. 5 (Amsterdam, 1960).

method ([30]) according to which,

$$\log \gamma_c = \frac{1}{n_s} \sum_{i=1}^{n_s} (\log \operatorname{cotg} \theta_i) , \tag{8}$$

n_s is the multiplicity of the shower particles. As can be seen from Table III, there is fairly good agreement between the values obtained by the two methods.

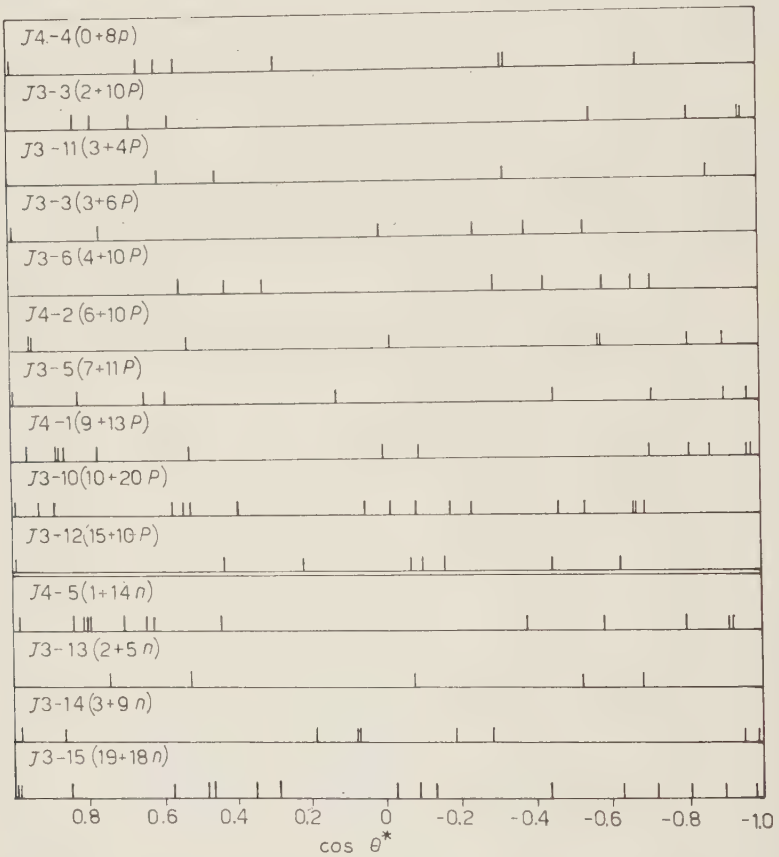

Fig. 5. – The angular distributions, in the C.M. system, of the shower particles in the secondary interactions.

The angular distributions in the C.M. system of shower particles produced in the secondary interactions are shown in Fig. 5. The angular distributions

([30]) C. Castagnoli, G. Cortini, C. Franzinetti, A. Manfredini and D. Moreno: *Nuovo Cimento*, **10**, 1539 (1953).

in the case of the secondary interactions produced by charged primaries do not appear to be different from those involving neutral primaries. Since the charged primaries are likely to be mostly pions and the neutral primaries non-pions, it seems that the angular distributions are independent of the nature of the primary particles (and in this context independent also of the P_T which should be different for the charged and the neutral primaries). At these energies, $\sim (10^{10} \div 10^{12})$ eV, the angular distribution seems to depend on N_h; with increasing N_h the distribution becomes more isotropic. This observation becomes apparent when one plots two separate histograms for events with $N_h \leqslant 6$ and $N_h > 6$. That this is so appears reasonable since an increase in N_h signifies, in general, an increase in the number of secondary collisions.

There is no indication of a « double-cone » structure in the F-plots of the secondary interactions except in one event *J*4-5, of tupe 1+14, due to a neutral primary; in this event the double-cone structure is very clear as can be seen from Fig. 6. The absence of the « double-cone » structure in these cases may be due to the fact that the primaries producing the secondary interactions are of relatively low energy; there is some indication that in nucleon-nucleon collision the « double-cone » structure becomes more apparent with increasing energy ([23]). A further possibility is that any « double-cone » structure which is present has been smeared out as a result of the secondary effect; in most of the secondary interactions high values of n_s are associated with high values of N_h and in these, secondary processes could very likely have occurred.

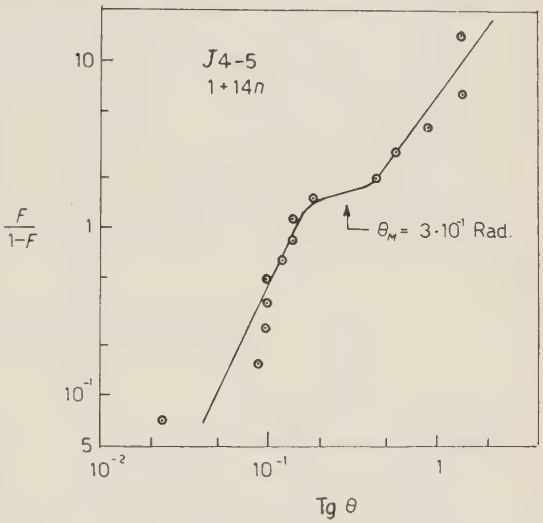

Fig. 6. – Plot of $[F/(1-F)]$ against tg θ for the secondary interaction *J*4-5, which is of the type 1+14n.

In view of these possibilities we cannot, on the available evidence, arrive at any conclusion on whether the absence of the « double-cone » structure in secondary interactions due to charged primaries can be taken as an indication that pion induced disintegrations will not exhibit such a structure. This would be an interesting point to pursue but would require for its answer much more experimental data.

4. – Momentum distribution of shower particles.

4·1. *The transverse momentum* P_T. – Recently, the transverse momentum P_T of shower particles produced in high energy interactions has been studied ([12,31]) in some detail. It has been found from these investigations that P_T has a reasonably peaked distribution with an average value $\sim (0.3 \div 0.5)$ GeV/c for π-mesons and $(1 \div 2)$ GeV/c for heavy particles. In Fig. 7

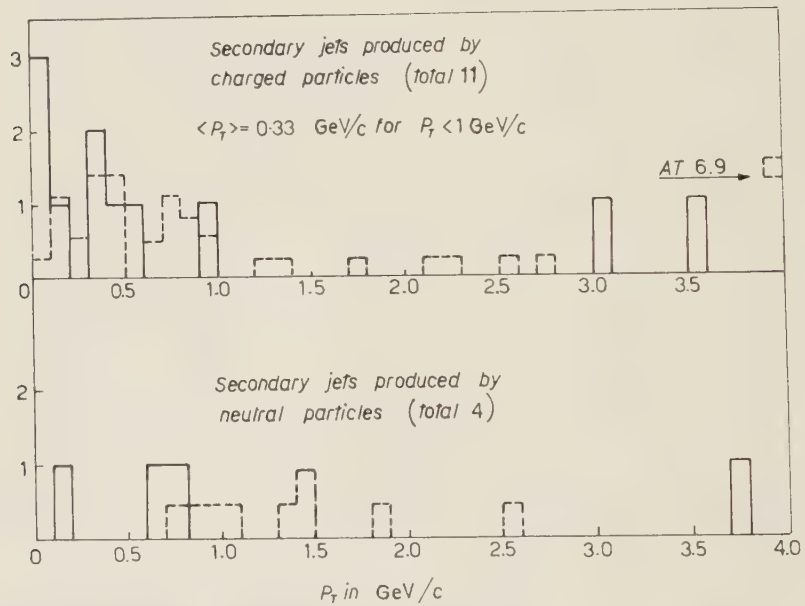

Fig. 7. – The transverse momentum, (P_T), distribution for the charged and neutral primaries of the secondary interactions. The dotted portions represent the results of the Bristol group; their data (36 charged and 9 neutral) have been normalized to the number of events observed by us.

we show our results together with the Bristol data ([12]) normalized to the number obtained by us. Within errors our results, which refer to nucleus-nucleus and nucleon-nucleus collisions, are consistent with the Bristol data, which refer primarily to nucleon-nucleon collisions.

([31]) O. MINAKAWA, Y. NISHIMURA, Y. TSUZUKI, H. YAMANOUCHI, H. AIZU, H. HASEGAWA, Y. ISHII, S. TOKUNAGA, Y. FUJIMOTO, S. HASEGAWA, J. NISHIMURA, K. NIU, K. NISHIKAWA, K. IMAEDA and M. KAZUNO: *Suppl. Nuovo Cimento*, **11**, 125 (1959).

4·2. *The momentum distribution in the L-system.* – If we now assume that P_T is constant and that the majority of the secondary particles are pions, then the angular distributions shown in Fig. 3 directly determine the momentum distributions of the shower particles in the L-system. As indicated on the curves, the differential momentum distributions are of the form $P^{-\alpha}\mathrm{d}P$ where $\alpha = 2.1$, 1.9, 2.1 and 2.3 for the shower particles in the inner cones of the jets *J*3, *J*4, *J*5 and *J*6 respectively. The distribution for the outer cone particles is much flatter, the exponent being $\alpha = 1.5$, 1.4, 1.4 and 1.3 for *J*3, *J*4, *J*5 and *J*6 respectively. This is in agreement with the results of NISHIKAWA [16] who analysed an event of energy $5 \cdot 10^{13}$ eV/nucleon, registered in an emulsion chamber and obtained $\alpha = 2.3$ and 1.5 for the inner and the outer cone particles respectively; in this event the nature of the primary and the size of the interaction are unknown.

Since the *J*5-jet indicates a well resolved « double-cone » structure, the observed angular distribution may be compared with the predictions of the « two-centre » model. If one assumes that the particles are emitted isotropically in the rest systems of the *lumps*, it can be shown that, for each of the two lumps taken separately, the integral cosec $\varphi (= P/P_T)$ spectrum of the shower particles in the L-system or the C.M. system of the collision is of the form

$$N(> \operatorname{cosec}\varphi = P/P_T) = \frac{n_s}{2}\gamma_b^2\left[(P/P_T)^2 + \gamma_b^2\right]^{-1}, \tag{9}$$

where n_s is the multiplicity of the event, φ is the angle of the shower particle in the system under consideration and γ_b is the Lorentz factor of the lump in that system. In deriving this formula we have assumed that $\beta_b \simeq \beta^{**} \simeq 1$ where β^{**} corresponds to the velocity of the particles in the lump system. Thus the P/P_T spectrum for *all* particles in the L-system is of the form

$$N(> P/P_T) = \frac{n_s}{2}\left[\gamma_{b1}^2\left\{(P/P_T)^2 + \gamma_{b1}^2\right\}^{-1} + \gamma_{b2}^2\left\{(P/P_T)^2 + \gamma_{b2}^2\right\}^{-1}\right], \tag{10}$$

where γ_{b1} and γ_{b2} are the Lorentz factors of the two lumps in the L-system In Fig. 3, the dotted curve shows the angular distribution or P/P_T spectrum according to (10) for the *J*5-jet, for which $\gamma_{b1} = 298$ and $\gamma_{b2} = 23$. As can be seen the experimental and the theoretical spectra are in fairly good agreement. If P_T is assumed to be constant, the above equations give directly the momentum spectra.

4·3. *The momentum distribution of shower particles in the C.M. system.* – The integral momentum spectra, in the C.M. system, of the inner cone shower

particles are plotted in Fig. 8 for the four jets. These are again of the form $P^{-\beta}\mathrm{d}P$, where $\beta = 2.3$, 1.9, 2.2 and 3.1 for $J3$, $J4$, $J5$ and $J6$ jets respectively It may be noted that the spectrum is steeper for the $J6$-jet whose primary energy is relatively low. However, the spectra for $J3$, $J4$ and $J5$ jets are con-

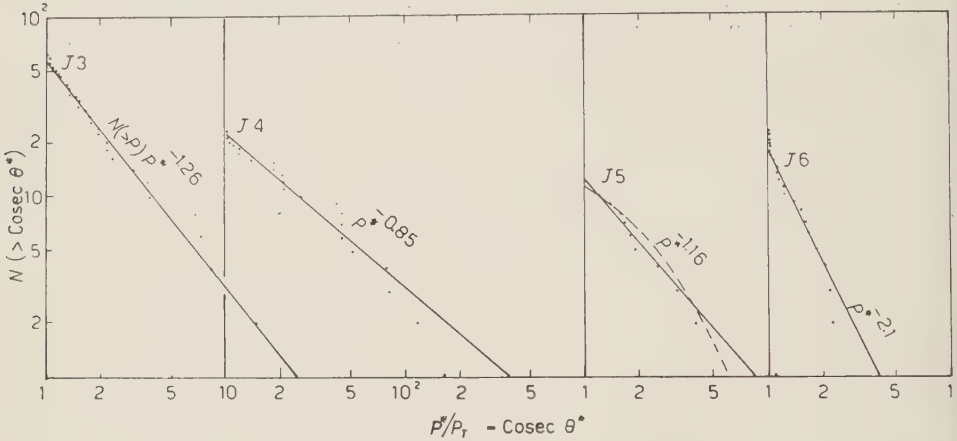

Fig. 8. – The integral angular distributions, in the C.M. system, for charged shower particles from the jets $J3$, $J4$, $J5$ and $J6$. The broken curve in the case of the $J5$-jet has been drawn on the basis of the « two-centre » model.

sistent with $P^{-2}\mathrm{d}P$, predicted by Heisenberg's theory ([32]). Our results are in disagreement with those of NISHIKAWA ([16]) who obtained $\beta = 2.5$ in the low and 4.1 in the high momentum regions for the event mentioned earlier. In the same figure (Fig. 8) the broken curve shows the integral momentum spectrum for the $J5$-jet ($\gamma^* = 1.8$) on the basis of the « two-centre » model. Again one can see that the theoretical spectrum is consistent with the experimental one.

5. – Conclusions.

We summarize below some features of nucleon-nucleus and nucleus-nucleus collisions at very high energy, obtained as a result of the present investigations on four jets $J3$, $J4$, $J5$ and $J6$ of the type $11+7+160$ C (of energy $8\cdot10^{13}$ eV/nucleon) $6+0+44$ p (of energy $8\cdot10^{13}$ eV/nucleon), $2+1+29\,\alpha$ (of energy $1.4\cdot10^{13}$ eV/nucleon) and $0+1+37\,N$ (of energy $1.2\cdot10^{12}$ eV/nucleon) respetively; the estimated energies of the jets are given within brackets.

([32]) W. HEISENBERG: *Zeits. Phys.*, **133**, 65 (1952).

i) A nucleon-nucleus event *J*4, and a nucleus-nucleus event *J*3, both at about 10^{14} eV/nucleon, show no « double-cone » structure of the type which has been observed in many examples of jets due to nucleon-nucleon collisions at ultra high energies ($\gtrsim 10^{13}$ eV/nucleon). It seems therefore, that at these energies, nucleus-nucleus and even nucleon-nucleus collisions, *involving high multiplicity*, do not indicate the « double-cone » structure.

ii) The angular distributions, in the L-system ,for the shower particles in the inner and the outer cones, are of the form $\mathrm{d}N \propto \mathrm{d}\theta$ and $\mathrm{d}N \propto \theta^{-0.6}\mathrm{d}\theta$ respectively, for nuclear interactions involving primary energies of ($10^{13} \div 10^{14}$) eV/nucleon.

iii) The angular distribution in the C.M. system, for nucleon-nucleus and nucleus-nucleus collisions at a primary energy $\sim 10^{14}$ eV/nucleon, is anisotropic, at least for primary nuclei of medium size; this is similar to the observations on nucleon-nucleon collisions. However, for one jet produced by a nitrogen nucleus of energy $\sim 10^{12}$ eV/nucleon, the angular distribution in the C.M. system does not show as strong an anisotropy as do the nucleon-nucleon collisions at this energy.

iv) The ratio of the number of secondary interactions produced by neutral particles to the number produced by charged particles is found to be 0.23 ± 0.15.

v) The transverse momentum distribution of the secondary shower particles produced in ultra high energy nucleon-nucleus and nucleus-nucleus collisions is consistent with the one observed for nucleon-nucleon collisions.

vi) At primary energies of ($10^{13} \div 10^{14}$) eV/nucleon the momentum spectrum, in the L-system, of the secondary shower particles in the inner cone is of the form $P^{-2}\,\mathrm{d}P$. In the C.M. system also, it has the same form *i.e.* $P^{-2}\,\mathrm{d}P$. The momentum spectra for the *J*5-jet, which indicates a « double-cone » structure, are also in good agreement with the « two-centre » model. But at a primary energy $\sim 10^{12}$ eV/nucleon, for one jet investigated, *J*6, the momentum spectrum in the C.M. system turns out to be much steeper.

It thus seems, from our observations on three jets, that at primary energies of ($10^{13} \div 10^{14}$) eV/nucleon the features of nucleon-nucleus and nucleus-nucleus collisions are similar to those of nucleon-nucleon collisions, except perhaps, for a suppression of the « double-cone structure » in the former.

* * *

We have great pleasure in expressing our thanks to Professor M. G. K. Menon and Dr. R. R. Daniel for valuable comments and discussions during the course of this investigation.

We wish to express our deep appreciation to the Office of Naval Research, U.S.A., who arranged for the exposure of the stack used in this experiment, on a balloon flight from Texas, and to Dr. F. B. McDonald for making detailed arrangement.

We are indebted to Mrs. T. M. Upadhyay and Miss F. Balsara who were responsible for finding most of the primary jets.

RIASSUNTO (*)

Si descrivono cinque interazioni nucleari prodotte da nuclei con valori di carica $Z = 1, 2, 6, 7$ e 14 e con energie fra 10^{12} e 10^{14} eV/nucleone. Si sono studiate le distribuzioni angolari e le distribuzioni di momento delle particelle secondarie prodotte in questi getti. Supposto che il momento trasversale sia costante, si trova che lo spettro del momento nel sistema del c.m. nel caso di ciascuno dei tre getti aventi energie fra $(10^3 \div 10^{14})$ eV/nucleone è della forma $P^{-2}\mathrm{d}P$. Si sono anche esaminate venti interazioni nucleari prodotte da particelle secondarie che sorgono in due dei getti, che coinvolgono energie di circa 10^{14} eV/nucleone. Il rapporto fra il numero di interazioni secondarie prodotte da particelle neutre e il numero di quelle prodotte da particelle con carica risulta essere 0.25 ± 0.15. Si sono determinate le energie delle particelle che provocano 16 delle interazioni secondarie. Si è riscontrato che la distribuzione del momento trasversale delle particelle secondarie, prodotte in collisione nucleone-nucleo nucleo-nucleo di 10^{14} eV/nucleone ed emesse con piccolo angolo (cioè, appartenenti al cono interno), corrisponde, nei casi qui studiati, a quella osservata in collisioni nucleone-nucleone di pari energia. Le osservazioni indicano che le collisioni nucleone-nucleo e nucleo-nucleo aventi un'energia di circa $(10^{13} \div 10^{14})$ eV/nucleone sono, in generale, simili alle collisioni nucleone-nucleone.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

On Symmetries Shared by Strong and Weak Interactions.

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(ricevuto il 17 Agosto 1960)

Summary. — A recent proposal due to Treiman (global symmetry and non-leptonic decay *js*-coupling) constitutes the starting point of this work. An inner inconsistency of the theory is noted: $\Sigma^+ \to n\pi^+$ is forbidden in the approximation employed. This may mean that the basic symmetry assumptions of the theory have to be discarded. On the other hand, there are two possibilities to avoid the inconsistency if need be, while at the same time the predictions on non-leptonic decays are maintained. *First possibility*: the relative sign ε of the $\Sigma\Sigma\pi$ relative to the π-nucleon interaction $= -1$. *Second possibility*: $\varepsilon = +1$. The *js*-interaction is replaced by a more general one (*js* and *jt* coupling). This coupling is subjected to a parity clash condition: *js* and *jt* separately are P-conserving, but together they are P-violating. As a consequence both reactions $\Sigma^\pm \to n\pi^\pm$ are P-conserving in the approximation of global symmetry. The clash condition is realized most simply if in the global symmetry approximation there exists only one type of S-violating current: either V or A. If it is assumed that the same S-violating currents occur in the leptonic as in the non-leptonic decays it then follows that the Ξ-nucleon parity has to be odd. The concept of parity clash is shown to be incompatible with an (overall current) $\times$ (overall current) structure for the totality of the weak interactions. It is incompatible with the idea that a vector boson-quartet carries all weak interactions. This possibility has to stand many tests (summarized at the end) to survive. In its most natural form it allows the reactions $\Sigma^+ \to n + e^+ + \nu$, $K^0 \to \pi^+ + e^- + \bar{\nu}$ to occur. In the simplest version of leptonic decays the charged lepton current itself plays a role analogous to the four vector potential in electromagnetism. All theoretical aspects of S-conserving decays remain unaffected.

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1. – Introduction.

Present evidence supports the view that for non-leptonic processes the $\Delta T = \frac{1}{2}$ rule ([1]) is valid to a good approximation. This rule cannot be rigorous in any case due to electromagnetism. The main experimental effect that indicates its approximate nature is the non-forbiddenness of $K^{\pm}_{\pi 2}$-decay. Throughout this paper we work in the approximation where the rule is supposed to be strictly valid for non-leptonic decays. We shall leave aside the question whether deviations from it are of purely electromagnetic origin.

For the leptonic S-violating processes there is only very limited information about the role of the $\Delta T = \frac{1}{2}$ rule. In this paper we shall not assume at the outset that the rule is necessarily valid also for these decays.

As is well-known ([2]), the $\Delta T = \frac{1}{2}$ rule establishes the following relation between the amplitudes of mesonic Σ-decays:

$$\langle \Sigma^+ | p\pi^0 \rangle \sqrt{2} = - \langle \Sigma^+ | n\pi^+ \rangle + \langle \Sigma^- | n\pi^- \rangle \,. \tag{1}$$

This relation is consistent with the present data. Several attempts have been made to establish theoretical links between these modes and the π-modes of the Λ by means of requirements stronger that the $\Delta T = \frac{1}{2}$ rule alone. Some of this work is phenomenological ([3]). Of other approaches we mention especially the ones by D'Espagnat and Prentki ([4]) and by Treiman ([5]) which lead to the same results. The first two authors assume essentially that the various amplitudes in question are subject to global symmetry conditions. The subsequent treatment due to Treiman ([5]), on the other hand goes much deeper, it constitutes a first attempt at an approximate dynamical theory. The main idea envisaged by him is that the weak interactions could possibly be probes of the symmetries of the strong interactions, provided the weak and the strong couplings share certain invariance properties. His work provides the starting point of the present considerations and we therefore state the Treiman conditions in detail.

a) The strong π-interactions satisfy global symmetry (G-symmetry). All results to be stated are derived in the approximation in which non-G-symmetric strong interactions are ignored.

([1]) M. Gell-Mann and A. Pais: *Proc. Conf. Nuclear and Meson Phys.*, Glasgow, 1954, p. 342 (London, 1955).

([2]) See *e.g.* M. Gell-Mann and A. Rosenfeld: *Ann. Rev. of Nucl. Sci.*, **7**, 454 (1957).

([3]) See *e.g.* S. Bludman: *Phys. Rev.*, **115**, 468 (1959).

([4]) B. d'Espagnat and J. Prentki: *Phys. Rev.*, **114**, 1366 (1959).

([5]) S. Treiman: *Nuovo Cimento*, **15**, 916 (1960).

b) The weak non-leptonic decay interactions are the product of an S-conserving $T=1$ current j and a $|\Delta S|=1$, $T=\frac{1}{2}$ current s, (js coupling). The coupling of j to s is such that their product transforms like $\Delta T=\frac{1}{2}$.

Remark. Until furter notice the space-time structure of j and s is immaterial.

c) The currents j, s each have a G-symmetric structure.

Remark. This statement really is too vague. It leaves for example an ambiguity in the definition of s. More concisely, as stated by TREIMAN [5] the weak and strong interactions are required to have certain particular invariance properties in common. For more on these points see Section **2**.

d) Final state interactions are neglected.

These conditions are sufficient to obtain the relation

$$\langle \Lambda | p\pi^- \rangle \sqrt{2} \approx \langle \Sigma^+ | n\pi^+ \rangle + \langle \Sigma^- | n\pi^- \rangle \,. \tag{2}$$

Here and in the following the sign $\approx$ denotes that we work with conditions and approximations like those stated in *a*)–*d*).

To see the important consequences of the combined equs. (1) and (2) we must first borrow the following information from experiment.

e) For the rates of the reactions $\Sigma^\pm \to n\pi^\pm$ we have to a good approximation

$$R(\Sigma^+ \to n\pi^+) = R(\Sigma^- \to n\pi^-) \,. \tag{3}$$

Furthermore, the amplitude $\langle \Sigma^+ | n\pi^+ \rangle$ is nearly pure S (or P). The parity mixture of $\langle \Sigma^- | n\pi^- \rangle$ is presently less well-known [6] but if we rely on the $\Delta T=\frac{1}{2}$ rule we may infer [2] that this amplitude should be nearly pure P (or S).

From *e*) and eqs. (1) and (2) it follows [3,4] that

$$R(\Sigma^+ \to p\pi^0) \approx R(\Lambda \to p\pi^-) \tag{4}$$

If one applies a phase space correction for the Σ, Λ-mass difference (which of course does not occur in the G-symmetry approximation) this relation becomes [5] $R(\Sigma^+ \to p\pi^0) \simeq 2R(\Lambda \to p\pi^-)$ which is in good agreement with ex-

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[6] For the Σ^+ experiments see R. L. COOL, B. CORK, J. W. CRONIN and W. A. WENZEL: *Phys. Rev.*, **114**, 912 (1959). The Σ^- results are due to P. FRANZINI, A. GARFINKEL, J. KEREN, A. MICHELINI, R. PLANO, A. PRODELL, M. SCHWARTZ, J. STEINBERGER and S. E. WOLF: *Bull. Am. Phys. Soc.*, **5**, 224 (1960).

periment. Secondly it follows from *e*) and eqs. (1) (2) that the large up down symmetry in $\Sigma^+ \to p\pi^0$ implies likewise a large up down asymmetry in $\Lambda \to p\pi^-$ again in agreement with experiment. Moreover, the prediction follows immediately ([4,5]) that

$$\alpha_0 \approx -\alpha_\Lambda \,, \tag{5}$$

where α_0, α_Λ are the longitudinal polarization of the proton in $\Sigma^+ \to p\pi^0$ $\Lambda \to p\pi^-$ respectively. As the proton in the Λ-reaction has left helicity ([7]) we should have right helicity for the Σ^+-mode. α_0 has thus for not been determined experimentally.

We cannot say whether this model is the only one which leads to such statements and predictions. Yet if equ. (5) were to agree with experiment we would have before us some very appealing conclusions. It should right away be emphasized that the contents of the present paper are likewise tied to the presumed validity of equ. (5). Thus if it would turn out that $\alpha_0 \approx \alpha_\Lambda$, all subsequent remarks would be of no practical interest.

It is also possible to deduce from *a*)-*d*) that ([4,5])

$$\langle \Xi^- | \Lambda\pi^- \rangle \approx \langle \Lambda | p\pi^- \rangle \,, \tag{6}$$

which provides a further test of the theory, in particular

$$\alpha_\Xi \approx \alpha_\Lambda \,. \tag{6a}$$

Treiman has also extended this theory to the leptonic hyperon decays. essentially by assuming that the charged part of the above mentioned *s*-current is also coupled to the charged lepton current. From this it follows ([5]) that

$$\langle \Sigma^- | n e^- \bar{\nu} \rangle \approx \sqrt{2} \langle \Lambda | p e^- \bar{\nu} \rangle \,. \tag{7}$$

It was also concluded ([5]) that there is a relation between the β-decay of Ξ^- and Λ. To this one can add that there exists likewise a connection between Ξ^0 and Λ β-decay. The results are

$$\begin{cases} \langle \Xi^- | \Lambda e^- \bar{\nu} \rangle \approx \langle \Lambda | p e^- \bar{\nu} \rangle \,, \\ \langle \Xi^0 | \Sigma^+ e^- \bar{\nu} \rangle \approx \sqrt{2} \langle \Lambda | p e^- \bar{\nu} \rangle \,. \end{cases} \tag{8}$$

([7]) E. Boldt, H. S. Bridge, D. O. Caldwell and Y. Pal: *Phys. Rev. Lett.*, **1**, 256 (1958). *Note added in proof*: see however R. Birge and W. Fowler: *Phys. Rev. Lett.*, 5, 254 (1960)

As has been emphasized by TREIMAN [5], the equs. (7), (8) offer many possibilities for experimental checks. There exist of course also similar relations for the μ-modes as well as for inverse neutrino processes. (The same remark applies to equs. (22)–(24) below). To conclude our summary, we recall that in this theory we must also have

$$\langle \Sigma^+ | n e^+ \nu \rangle = 0 , \qquad \langle \Xi^0 | \Sigma^- e^+ \nu \rangle = 0 , \tag{9}$$

$$\langle K^0 | \pi^+ e^- \bar{\nu} \rangle = 0 , \tag{10}$$

as here the $\Delta T = \frac{1}{2}$ rule also applies to the leptonic decays.

The investigations which form the subject of the present paper likewise make ample use of G-symmetry considerations. It is perhaps not necessary to repeat here all the arguments [8] which show that if G-symmetry actually holds true in nature, this has certainly been carefully masked till now. All that matters for the present is, first that G-symmetry has neither been proved nor disproved thus far, secondly that if it were so that relations like equs. (5),(6) would turn out to be in agreement with experiment, it might be useful to pursue the possibility, even though not uniquely implied, that weak interactions probe symmetries of strong interactions by sharing these symmetries to some extent. There is presently so little to go on by way of exploring the laws of particle dynamics that every line of approach which offers some means of confrontation with experiment must be pursued to its logical conclusion.

As a matter of fact the result which forms the point of departure of this paper seems at first sight to be a heavy blow against the conclusions summarized above. It will namely be proved in Section **2** that the assumptions *a*)–*d*) do not only imply equ. (2) but at the same time also

$$\langle \Sigma^+ | n \pi^+ \rangle \approx 0 , \qquad (js\text{-coupling}) . \tag{11}$$

or, a little more precisely, the ratio $\langle \Sigma^+ | n \pi^+ \rangle / \langle \Sigma^+ | n \pi^- \rangle$ vanishes in the G-approximation. In other words, the theory itself is inconsistent with the experimental input *e*) above.

It may well be argued that equ. (11) means actually very little. Other non-zero physical quantities are known to vanish in the G-approximation [9] and one may say that « corrections » to G-symmetry can be so large as to eliminate the contradiction with equ. (11). However, we shall see that equ. (11) is derived by exactly the same kind of argument as equ. (2). Thus if one

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[8] See *e.g.* A. PAIS: *Phys. Rev.*, **110**, 574, 1480 (1958).

[9] For example the amplitude for $\pi_0 \to 2\gamma$, see R. PUGH: *Phys. Rev.*, **109**, 989 (1958).

disavows equ. (11) one must do the same with equ. (2) and all the interesting relations between Σ- and Λ-decay stated above would then be meaningless. So one would have to conclude that the symmetries of the strong interactions once again continue their masquerade!

(Remark. The above is of course no argument against a *js*-structure of the weak non-leptonic interaction, though it could be an argument against the G-symmetric form of the currents j and s).

While this negative view may turn out to be the correct one, it seems worth while nevertheless to state that there are at least two possibilities which may, if need be, save the relations (2)–(6).

The first possibility. As was observed by GELL-MANN [10], there are really two physically distinct versions of G-symmetry. They are characterized by the value $\varepsilon = \pm 1$ of the relative strength of the $\Sigma\Sigma\pi$-versus the π-nucleon interaction. This sign has a physical relevance of a similar kind as, say, the relative sign of π^0 coupling to protons as compared to neutrons. For what follows it is essential to state that the G-symmetry operations for $\varepsilon = +1$ are distinct from those for $\varepsilon = -1$ (for details see Section **2**). We therefore propose to keep these symmetries apart by a different symbol for each case:

(12) $\qquad \varepsilon = \pm 1$ corresponds to $G^{\pm}$-symmetry.

What we have heretofore called G-symmetry is actually G^{+}-symmetry in the present terminology. It will be shown in Section **2** that the relation (2) is in essence independent of whether $\varepsilon = +1$ or -1. On the other hand we shall also prove that equ. (11) is one of the two distinct alternatives

(13) $\qquad \langle \Sigma^{+} | n\pi^{+} \rangle \approx -\varepsilon \langle \Sigma^{+} | n\pi^{+} \rangle$, ($js$-coupling, $G^{\pm}$-symmetry),

which is catastrophical for $\varepsilon = +1$ and trivial for $\varepsilon = -1$. Thus our first possibility is: the symmetry of the strong interactions is G^{-}-symmetry in which case there is no inconsistency between the equations (1)–(5). Also the relations (7)–(10) remain acceptable. This case is briefly commented on in Section **3**.

Remark. It will be emphasied in Section **2** that the distinction between G^{+}- and G^{-}-symmetry can be formulated entirely independently of the question whether the π-baryon interaction is of such a specific form as for example $\bar{\psi}\boldsymbol{\tau}\gamma_5\psi\boldsymbol{\pi}$. This seems worth mentioning as there is nothing sacred about this form, as far as is presently known.

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[10] M. GELL-MANN: *Phys. Rev.*, **106**, 1296 (1957).

The second possibility. There exists a second point of view with far reaching implications which are physically distinct from the previously mentioned case of G^--symmetry. Here one retains G^+-symmetry and considers equ. (11), (or equ. (13) with $\varepsilon = +1$) rather as a blessing in disguise.

The physical idea is the following. The near parity-conserving properties of the $\Sigma^\pm \to n\pi^\pm$ modes are one of the most challenging pieces of experimental information on hyperon decays that we know of. It may be that this is somewhat of a dynamical accident. On the other hand, it does not seem entirely too far-fetched to contemplate the possibility that this information points, in itself, to some approximate invariance property shared by weak and strong interactions. Without prejudice as to whether this last view is correct, let us accept it for the moment as a working hypothesis. Then, even apart from the troublesome equation (11), there is something unsatisfactory in the state of affairs outlined above. It seems then a little peculiar to make invariance assumptions like *a*)–*c*) and at the same time to use in *e*) the parity-conserving features of $\Sigma^\pm \to n\pi^\pm$ simply as experimental information. One would then rather, if possible, attempt to rephrase the conditions in such a way that not only the interesting results about the relation between $\Sigma^+ \to p\pi^0$ and $\Lambda \to p\pi^-$-modes but also these striking properties of the other two Σ-modes appear as a consequence. We shall show that this can indeed be done, thanks precisely to equ. (11). An understanding of the near parity-conservation in $\Sigma^\pm \to n\pi^\pm$ by means of invariance arguments now becomes the central theme of the second possibility.

Before stating how this comes about, we would like to raise a further question. Could one not with equal right consider the relation (3) as something « deep » rather than as a dynamical accident? There is little doubt that the answer is no. On the basis of the $\Delta T = \frac{1}{2}$ relation, equ. (3) represents a near equality between an S-wave and a P-wave reaction; there is no conceivable reason why such an equality could have anything to do with symmetry arguments. On the other hand, this near equality may turn out to be not so surprising once it will be understood why parity is nearly conserved in these reactions. In this connection a suggestive result due to BLUDMAN [3] may be recalled. This author obtains to a good approximation the relation (3) by assuming first that one mode goes via a pure V-, the other via a pure A-coupling where the ratio of A to V strengths is chosen to be $\sim 1, 2$, as is suggested from β-decay.

We shall now state in a little more detail the second possibility to prevent that equ. (11) constitutes a paradox. First the question was asked: once we realize which invariance properties the strong and weak interactions have to share to arrive at equ. (2), is it a necessary consequence of these properties that the weak non-leptonic interaction has the *js*-structure? It turned out that the answer is no, and that there exists a second class of couplings which

likewise leads to equ. (2). These couplings will be called *jt*-couplings. They are described in detail in Section **4**. In essence, the *t*-current is a G-symmetric structure which contains $T=\frac{1}{2}$ and $\frac{3}{2}$; and *j*, *t* couple together in a simple form which guarantees once again $\Delta T=\frac{1}{2}$. Moreover for *jt*-coupling equ. (11) does not hold true, so we get a non-vanishing $n\pi^+$-amplitude. But, in turn, the *jt*-coupling has a remarkable property, namely

$$\langle \Sigma^- | n\pi^- \rangle \approx 0\,, \quad (jt\text{-coupling})\,. \tag{14}$$

Therefore one arrives at the following picture for a combined (*js*) *and* (*jt*) coupling: (*js*) contributes to $n\pi^-$ but not to $n\pi^+$; (*jt*) contributes to $n\pi^+$ and not to $n\pi^-$; while it furthermore is found that both *js and jt* contribute to the strongly *P*-violating $\Sigma^+ \to p\pi^0$ mode and to the Λ-modes. Moreover any linear combination of (*js*) and (*jt*) yields equs. (2) and (6).

I was therefore led to consider the following possibility: the *js*- and *jt*-couplings are separately parity conserving but in such a way that the combined (*js*, *jt*) interactions violate parity. This idea of a parity clash guarantees that to our approximation the $\Sigma^\pm \to n\pi^\pm$ modes are *P*-conserving while $\Sigma^+ \to p\pi^0$, $\Lambda \to p\pi^-$ violate parity. If one now further *assumes* that equ. (3) holds approximately true, we are then guaranteed by previous arguments that $\Sigma^+ \to p\pi^0$, $\Lambda \to p\pi^-$ are nearly maximally *P*-violating. To prove that equ. (3) holds true on the basis of the envisaged dynamics has so far not been feasible; on the other hand the possibility cannot be excluded that further arguments would build a bridge between this theory and Bludman's just mentioned phenomenological interaction ([3]). If this were to succeed, the argument would be fairly complete.

Continuing this line of thought, the next question is: how can one make *js* and *jt* separately *P*-conserving but in such a way that they clash when taken together? It will be shown in Section **5** (*a*) that there are essentially two ways of doing this: either *j* has the same « pure » space time structure (*e.g.* either *V* or *A*) in both *js* and *jt*; while *s* and *t* have different reflection properties, for example *s* is *A*, *t* is *V*; or *s* is *V*, *t* is *A*. The other way is to assume that the *j* occurring in *js* has different reflection properties (*e.g. A*) than the *j* occurring in *jt* (*e.g. V*), while *s* and *t* have the same reflection characteristics.

If we are at all going in the right direction, this last possibility is more attractive: we know for certain, from the *S*-conserving processes, that both a *V*- and an *A*-current of the *j*-type do occur in the weak interactions. For the *S*-violating currents no such definite knowledge exists. Thus it now seems simplest to assume:

f) The *S*-violating currents are « pure », that is, they are either all of the *V* (*i.e.* γ_λ) type or all of the *A* (*i.e.* $\gamma_\lambda\gamma_5$) type. In other words: either there

does *not* exist an S-violating vector (or rather γ_λ) current, or there does *not* exist an S-violating axial vector (or rather $\gamma_\lambda\gamma_5$) current. In the discussion below of hyperon β-decay, it will be observed that if the assumption *f*) were to be true, the case of a V-type S-violating current is preferable. If so, there does not exist an axial vector (or rather $\gamma_\lambda\gamma_5$) current.

Remark. In denoting currents by γ_λ or $\gamma_5\gamma_\lambda$ we refer of course to their baryonic parts. The existence of a mesonic part is nowhere precluded. As will be seen in Section **5**-*b*) it is definitely a better terminology to speak of γ_λ rather than of vector currents when strong K-interactions are also included. As we shall see, this language is in fact necessary if the presently unknown ([11]) cascade parity would turn out to be odd.

If we accept all this, there still remain two alternatives which, in obvious notation, can be designated either ($j^A s^V$, $j^V t^V$) ot ($j^V s^V$, $j^A t^V$). These possibilities are physically distinct, as can be seen as follows: the reaction $\Sigma^+ \to n\pi^+$ proceeds, (always to our approximation) via *jt*-coupling only. Thus the corresponding amplitude is P-wave for $j^V t^V$; it is S-wave for $j^A t^V$.

In order to determine experimentally the nature of the *j*-current which couples to t^V one notes the following. Let Σ^+ be transversely polarized in production, with a polarization vector $\boldsymbol{\Pi}_\Sigma$ in its rest frame. Denote by $\boldsymbol{n}$ the unit vector in the direction of the line of flight of the decay neutron in the same frame. The polarization vector of the neutron in its own rest frame will be called $\boldsymbol{\Pi}_n$. Then we have, to the approximation that possible small P-violations in the decay $\Sigma^+ \to n\pi^+$ are neglected,

$$\Sigma^+ \to n\pi^+ \text{ via} \begin{cases} S\text{-wave:} & \boldsymbol{\Pi}_n = \boldsymbol{\Pi}_\Sigma\,, \\ P\text{-wave:} & \boldsymbol{\Pi}_n = 2(\boldsymbol{\Pi}_\Sigma \boldsymbol{n})\cdot \boldsymbol{n} - \boldsymbol{\Pi}_\Sigma\,. \end{cases} \tag{15}$$

The next question concerns the relative weight of the *js* and the *jt* couplings. It turns out that essentially equal weight implies that the $\overline{\Lambda}p$ term is coupled to the V-A combination ($j^A + j^V$). This relative weight is tentatively assumed here. We refer to Section **5**-*a*) for comments on the possible relation between the dynamical non-leptonic decay coupling so obtained and the phenomenological non-leptonic decay interaction given by Bludman ([3]).

This concludes the remarks on the non-leptonic modes. We note that the present treatment shares with earlier ones ([4,5]) the unpleasant feature that no estimate is given of the « corrections » to the various relations due to those strong interactions which violate G-symmetry.

([11]) See however D. H. Wilkinson, S. J. St. Lorant, D. K. Robinson and S. Lokanathan: *Phys. Rev. Lett.*, **3**, 397 (1959), footnote ([7]).

Next we must ask what are the consequences of our second possibility of the leptonic S-violating decays. Here it is necessary first of all to discuss the implications with respect to the universal version of weak interactions. Let us briefly review the situation.

The following attractive possibility ([12]) has been much discussed in recent years. There exists an « overall current », additively composed of S-conserving and S-violating baryon parts, possibly of meson parts and of a leptonic part. All weak interactions are deduced from a coupling which may briefly be denoted as

(16) Total weak interaction = (overall current) × (overall current) .

In this picture, the various weak interactions, leptonic and non-leptonic, are due to cross-terms between different composing parts of the overall current. Moreover the structure (16) implies, through its « pure square » terms, additional effects such as weak electron-neutrino scattering, weak (parity violating) baryon-baryon interactions, etc.

In its original version the interaction (16) was supposed to involve charged currents only. It was soon recognized, and first stated by SAWYER ([13]) for the case of non-leptonic decays, that the $\Delta T = \frac{1}{2}$ rule is only then compatible with equ. (16) if also neutral overall currents are admitted. However, this raised a new question: when we include leptonic processes, should the neutral overall current have a lepton part J_0? This would be expected, if (16) is correct, on grounds of symmetry. It is true, as noted by BLUDMAN ([3]), that certain S-conserving weak processes due to J_0-coupling would be extremely hard to detect. On the other hand, the presence of J_0 in conjunction with a structure (16) seems to pose real difficulties with respect to S-violating processes ([14]). Unless further experimental and theoretical arguments change the picture, it seems proper to speak of « the mysterious absence of neutral lepton currents ». No light is shed on this mysterious absence by the proposal of LEE and YANG ([14]) that all weak interactions are mediated by a quartet of vector bosons W which are semi-weakly coupled to the overall currents in such a fashion that equ. (16) emerges as an effective second order interaction.

Finally, we recall that the structure (16) also imposes an important restriction on the S-violating charged currents: it is not allowed that this part of the current contains both a $\Delta S/\Delta Q = 1$ part and a $\Delta S/\Delta Q = -1$ part, be-

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([12]) See R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1957); E. SUDARSHAN and R. MARSHAK: *Phys. Rev.*, **109**, 1860 (1957).

([13]) R. SAWYER: *Phys. Rev.*, **112**, 2135 (1958).

([14]) T. D. LEE and C. N. YANG: *Phys. Rev.*, **119**, 1410 (1960).

cause their interference would produce a weak $\Delta S = 2$ interaction. OKUN and PONTECORVO ([15]) have given a stringent theoretical condition on the $K_1 - K_2$ mass difference which would exclude the presence of such interactions with a strength comparable to other weak couplings; and this condition is well met experimentally ([16]).

We now confront two ideas with each other. The first one is that the characteristics of the non-leptonic Σ-decay modes are due to a parity clash in the sense discussed above. However, we leave entirely open for the moment whether this parity clash is correctly described by our (js), (jt) couplings. The second idea is that the total weak interaction has the structure (16). We show that these two ideas are incompatible.

This is obvious. The overall current should be of the form $j_A + j_V + C + \ldots$, where $j_{A,V}$ are tye A, V parts of the S-conserviag current and C is the S-violating current. The coupling of the form (16) then implies that C is coupled to both j_A and j_V, contrary to the possibility of parity clash. This argument is also independent of the various alternatives mentioned just before assumption (f).

Remark. This reasoning would fails if the parity clash were one between charged current interactions on the one hand, neutral current interactions on the other. However, such a type of clash would violate the $\Delta T = \frac{1}{2}$ rule.

Returning to the specific (js), (jt)-coupling we obtain a further argument in this direction. It was noted by TREIMAN ([5]) that s is of the $\Delta S/\Delta Q = +1$ type. However, we shall see in Section **4** that t contains both $\Delta S/\Delta Q = \pm 1$. In view of the $\Delta S = 2$ argument just mentioned, we may not couple s to t or t to itself.

While all this does not preclude the notion of a universal strength of weak interactions, we must however retreat from the universal form (16). This may turn out to be a very serious drawback to the notion of parity clash in general and to the (js), (jt)-coupling scheme in particular. Specifically, the existence of a W-quartet which carry all weak interactions is excluded by any such theory. Thus the discovery of W-particles with all the properties discussed by LEE and YANG ([11]) would automatically eliminate this possibility.

Let us grant that the disadvantage of renouncing on the form (16) may be calamitous. Let us nevertheless ask by what criteria we may now approach anew the question of the overall structure of the weak interactions on the basis of the parity clash idea. To this end we first return to the question of the neutral lepton currents. We have seen that this question arose in an

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([15]) L. OKUN and B. PONTECORVO: *Žurn. Éksp. Teor. Fiz.*, **32**, 1587 (1957); translation *Journ. Exp. Theor. Phys.*, **5**, 1297 (1957).

([16]) F. MULLER, R. W. BIRGE, W. B. FOWLER, R. H. GOOD, W. HIRSCH, R. P. MATSEN, L. OSWALD, W. M. POWELL, H. S. WHITE and O. PICCIONI: *Phys. Rev. Lett.*, **4**, 418 (1960).

attempt to reconcile the structure (16) in the most natural way with the $\Delta T = \frac{1}{2}$ rule. Now that we anyway have to contemplate the abandonment of equ. (16), it will be clear that the next best statement is the following. The weak interactions consist of two additive parts, the non-leptonic interactions which do satisfy $\Delta T = \frac{1}{2}$ and the leptonic interactions which do not necessarily obey this rule. The latter are of the general form of charged currents coupled to each other. We denote this as follows:

(17) Total weak interactions =

$$= \text{Non-leptonic interactions } (\Delta T = \tfrac{1}{2}) + \text{leptonic interactions}\,.$$

Of course it would at the present stage be an overstatement to consider it as an argument in favor of the present version that the question of the absence of J_0 simply never arises.

Next we ask, what is the simplest form of the leptonic decay interaction in this picture? Observe that as a consequence of equ. (17) we have yet to specify which S-violating currents mediate S-violating leptonic processes. We assume

g) In the leptonic S-violating decay interactions no other S-violating currents intervene than the currents s and t mentioned above.

Of course this is less compelling than it is in a theory where equ. (16) holds true, but it seems a natural step to take. This assumption has an important physical consequence. In Section **5**-*b*) the following is shown.

The assumptions *f*) and *g*) imply that the Ξ-nucleon parity must be odd as long as parity is strictly conserved in strong interactions.

As will be seen in Section **5**-*b*), this statement follows from a comparison of the structure of the K^{+}_{e2} and the K^{+}_{e3} amplitudes. There it will also be shown that the inference on the Ξ-nucleon parity is independent of the choice γ_λ of $\gamma'_\lambda\gamma'_5$ for the baryonic (s, t)-currents. Note that odd Ξ-nucleon parity is incompatible with the Salam-Polkinghorne classification of particles ([17]).

There exists at present only limited dynamical information on the leptonic S-violating decays. However there is good evidence ([18]) that the lepton current involved in K-decays is as predicted by the two component neutrino theory. Accordingly we shall assume that the same lepton current J of the

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([17]) A. Salam and L. Polkinghorne: *Nuovo Cimento*, **2**, 685 (1955). Here N_1 and N_4 are united to the representation $(\frac{1}{2}, \frac{1}{2})$ in the presence of all strong interactions.

([18]) C. A. Coombes, B. Cork, W. Galbraith, G. R. Lambertson and W. A. Wenzel: *Phys. Rev.*, **108**, 1348 (1957).

V-A type is responsible for all leptonic processes. Its charge components will as usual be denoted by $J_\lambda^\pm$. Thus it would seem that the simplest form of the leptonic interaction in equ. (17) is

(18) Leptonic interaction =

$$= (\text{overall charged current}) \times (\text{leptonic charged current}) .$$

Remarks.

1) Using the oft made analogy with electrodynamics, it is $J_\lambda^\pm$ itself which here plays a role similar to the electromagnetic potential A_λ. The overall currents of course contain $J^\pm$ as well. One may call the $J_\lambda^+ J_\lambda^-$ terms in (18) the analog of the electromagnetic A_λ^2-terms, familiar from boson couplings.

2) There is of course no objection to the existence of weak P-violating nucleon-nucleon interaction due to $(j^A + j^V)^2$-terms, in our actual state of knowledge. On the other hand, from the present point of view there seems be no compelling reason to include such terms.

3) A structure like equ. (18) does not exclude the possibility that weak interactions are mediated by more complex systems of intermediate bosons than just the W-quartet, but for the present this does not seem a very attractive possibility to pursue.

4) The interaction (18) shares with (16) the need for arguments which show for example why renormalization effects do not affect the strength of the S-conserving vector current relative to the lepton current. The idea of a conserved S-conserving vector current advanced by FEYNMAN and GELL-MANN [12] and pursued by GELL-MANN [19] is of course applicable without modification for the interaction (18).

5) As was stated before, we need never bother now about the neutral lepton current J_0. In this connection it should be noted that an entirely different coupling scheme exists where J_0 cannot intervene. This is the scheme due to D'ESPAGNAT [20]. As this author notes, it is also possible to put his scheme into a form where the effective currents satisfy G-symmetry. It is however readily verified [21] that, even so, the scheme does not yield $\Sigma\Lambda$- or $\Xi\Lambda$-relations of the type of equs. (2), (6). This in itself is not necessarily an argument against this approach [22] which could become of particular interest

[19] M. GELL-MANN: *Phys. Rev.*, **111**, 362 (1958).

[20] B. D'ESPAGNAT: *Nuovo Cimento*, **18**, 287 (1960).

[21] B. D'ESPAGNAT: private communication.

[22] As always in such considerations, the fact that one cannot prove a relation on general grounds does not mean that the same relation couldn't hold true for detailed dynamical reasons.

if it were found that cascade particles decay leptonically into nucleons. This phenomenon is a natural, though not strictly necessary, implication of d'Espagnat's scheme. In the present case both s and t satisfy $\Delta S = 1$ so that no such β-decay of the Ξ can occur.

Finally there arises the question of the relative weight with which s and t occur in the overall current in equ. (18). It is logically possible to couple s, not t, to J but this seems artificial. We consider then the case where both s and t are coupled to J and note the following.

1) This implies that

$$\Sigma^+ \to \mathrm{n} + \mathrm{e}^+ + \nu\,, \qquad \Xi^0 \to \Sigma^- + \mathrm{e}^+ + \nu\,, \tag{19}$$

$$\mathrm{K}^0 \to \pi^+ + \mathrm{e}^- + \bar{\nu} \tag{20}$$

are allowed, as t has a $T = \frac{3}{2}$ part.

2) The assumption that $\Delta T = \frac{1}{2}$ also in leptonic decays implies ([23])

$$\langle K^+ | \pi^0 e^+ \nu \rangle \sqrt{2} = \langle K^0 | \pi^- e^+ \nu \rangle\,, \qquad (\Delta T = \tfrac{1}{2}). \tag{21}$$

The present experimental information ([24]) is not yet sufficiently precise to judge the quantitative correctness of this important prediction.

3) It will be shown in Section 5-*b*) that instead of equs. (7), (8) one now has

$$\langle \Lambda | p e^- \bar{\nu} \rangle \sqrt{2} \approx \langle \Sigma^- | n e^- \bar{\nu} \rangle + \langle \Sigma^+ | n e^+ \nu \rangle\,, \tag{22}$$

$$\langle \Xi^- | \Lambda e^- \bar{\nu} \rangle \approx \langle \Lambda | p e^- \bar{\nu} \rangle\,, \tag{23}$$

$$\langle \Lambda | p e^- \bar{\nu} \rangle \sqrt{2} \approx \langle \Xi^0 | \Sigma^+ e^- \bar{\nu} \rangle + \langle \Xi^0 | \Sigma^- e^+ \nu \rangle\,. \tag{24}$$

We shall see that these relations are independent of the numerical value of the ratio of s to t coupling strength to J. (Of course certain amplitudes vanish for zero t-coupling). Relations like (22)–(24) can of course easily be corrected for phase space differences ([5]).

([23]) S. Okubo, R. E Marshak, E. C G. Sudarshan, W. B. Teutsch and S. Weinberg: *Phys. Rev.*, **112**, 665 (1958).

([24]) F. S. Crawford, M. Cresti, R. L. Douglas, M. L. Good, G. R. Kalbfleisch and M. L. Stevenson: *Phys. Rev. Lett.*, **2**, 361 (1959).

4) If the role of induced terms ([25]) in hyperon β-decay is negligible and if renormalization would not affect the s to t ratio we can say more, namely: if s and t enter with equal weight then (see equ. (64) below)

$$\langle \Sigma^- | ne^- \bar{\nu} \rangle \approx \langle \Sigma^+ | ne^+ \nu \rangle \approx \frac{1}{\sqrt{2}} \langle \Lambda | pe^- \bar{\nu} \rangle \,. \tag{25}$$

Note that equ. (23) gives Σ-decay rates with a strength smaller by a factor two than the Λ-rate, while equ. (7) gives an effect larger by a factor two.

5) Present knowledge on hyperon β-decay is exceedingly scanty ([26]). If one thing stands out it is the surprising smallness of the decay rates which represents rather a puzzle on the basis of the universal interaction picture (16) where all baryon currents are of the V-A type. While it is certainly difficult to forget that the universal interaction (16) has weathered many storms, it may nevertheless be pointed out that the present alternative sheds perhaps some light on this problem. Let us recall that in our second possibility the S-violating currents are pure. Now for the ratio of decay rates of a 1:1 V-A mixture and a pure V-interaction one has in the non-relativistic limit

$$\frac{R(V)}{R(V\text{-}A)} = \frac{1}{4} \,. \tag{26}$$

This relation remains true to about 1 percent for the actual Λ and Σ kinematics ([27]). If we take a V-1.2A mixture and compare that with pure V, we get $R(V)/R(V\text{-}1.2A) = 1/5.3$. This leads us to suggest that if the picture of a pure S-violating current is correct, the smallness of hyperon β-decay may find to a considerable extent its interpretation by the choice of a pure V (or rather γ_λ) current.

Of course there remain many questions such as the relative weight of S-conserving versus S-violating currents and in particular the influence of K-interactions on relative renormalizations. About these largely dynamical problems we have nothing to say in this paper, beyond one qualitative remark in Section **5**-*b*) about the influence of K-couplings on the parity-purity of the s, t-currents. As will be seen in the summary given in Section **5**-*c*), there are

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([25]) C. Albright: *Phys. Rev.*, **115**, 750 (1959).

([26]) For the status of hyperon β-decay see L. Okun: *Ann. Rev. of Nucl. Sci.*, **9**, 82 (1959).

([27]) See *e.g.* Y. Yamaguchi: *Table of the leptonic decay rate of the hyperon* (CERN Report 59-18, 1959).

many ways in which the second possibility could be disproved by experiment. Even in the event that all such disproofs were to fail it would perhaps not be easy to say that this possibility is correct in all the detail in which it is described here.

In conclusion we state what are the broadest aspects of the present approach to strong and weak interactions. It is the hope that we may uncover some of the strong interaction symmetries and parities if it were correct that K-interactions do not strongly distort the non-leptonic and leptonic decays of hyperons; and that shared symmetries of strong and weak interactions provide a main theoretical tool for this purpose. Even if this hope were to be fulfilled there remain many questions of which two stand out. The first one is the way in which presumed strong interaction symmetries transcending charge independence get broken down, as they have to [8]. Related to this is the question of the mass splits in the particle spectra. The second is, how does one understand the universality of strength of all weak interactions in case it were necessary to modify the universal form (16). We need more experimental background before we can say that either one or both of these questions are the right ones to face.

2. – G$^{\pm}$-symmetry and *js*-coupling.

We ask for those invariance properties to be shared by that part of the strong interactions which satisfies G$^{\pm}$-symmetry on the one hand and by the weak non-leptonic interactions on the other and which are sufficient to derive equ. (2). For the purpose of exposition we assume that the strong interactions in question are the trilinear baryon-baryon-couplings. Once we shall have established the sought for symmetry properties the precise form of those strong interactions becomes rather immaterial as long as the invariance is satisfied.

We introduce the following notation [8]

$$(27) \qquad N_1 = \begin{pmatrix} p \\ n \end{pmatrix}, \qquad N_2 = \begin{pmatrix} \Sigma^+ \\ Y^0 \end{pmatrix}, \qquad N_3 = \begin{pmatrix} Z^0 \\ \Sigma^- \end{pmatrix}, \qquad N_4 = \begin{pmatrix} \Xi^0 \\ \Xi^- \end{pmatrix},$$

$$Y^0 = \frac{\Lambda - \Sigma^0}{\sqrt{2}}, \qquad Z^0 = \frac{\Lambda + \Sigma^0}{\sqrt{2}},$$

and write the strong π-interaction as

$$(28) \qquad H_s = iG\,[\bar{N}_1 \boldsymbol{\tau} \gamma_5 N_1 + \varepsilon(\bar{N}_2 \boldsymbol{\tau} \gamma_5 N_2 + \bar{N}_3 \boldsymbol{\tau} \gamma_5 N_3) + \bar{N}_4 \boldsymbol{\tau} \gamma_5 N_4]\,\boldsymbol{\pi}\,,$$

with ε defined by equ. (12). Following TREIMAN [5] we assume the weak non-

leptonic decay interaction to be (apart from a constant)

$$\text{(29)}\qquad \begin{cases} H_w(js) = H_w^{\mathrm{ch}}(js) + H_w^0(js)\,, \\ H_w^{\mathrm{ch}}(js) = j^+ s^- + j^- s^+\,, \\ H_w^0(js) = -\dfrac{1}{\sqrt{2}}\, j^0 (s^0 + \bar{s}^0)\,. \end{cases}$$

Here

$$\text{(30)}\qquad \begin{cases} j^\pm = \dfrac{1}{\sqrt{2}}\,(j_1 \pm i j_2)\,, \qquad j^0 = j_3\,, \\ \boldsymbol{j} \;= \sum_{i=1}^{4} \bar{N}_i \boldsymbol{\tau} N_i\,. \end{cases}$$

Thus $\boldsymbol{j}$ is an S-conserving $T=1$ vector. The S-violating $T=\frac{1}{2}$ spinor components s are given by [5]

$$\text{(31)}\qquad s = \begin{pmatrix} s_- \\ s_0 \end{pmatrix} = \begin{pmatrix} \bar{N}_3 N_1 + \bar{N}_4 N_2 \\ \bar{N}_2 N_1 - \bar{N}_4 N_3 \end{pmatrix}.$$

For the purposes of this section we need not specify the space time properties (V and/or A or other possibilities) of j and s.

We need two symmetry properties to establish equ. (2). The first one is

$$\text{(32)}\qquad N_1 \leftrightarrow N_2\,, \qquad N_3 \leftrightarrow N_4\,, \qquad \boldsymbol{\pi} \rightarrow \varepsilon \boldsymbol{\pi}\,,$$

which leaves H_s and $H_w(js)$ invariant provided the permuted baryon doublets are mass degenerate. From (32) it follows that $\langle \Upsilon^0 | p\pi^- \rangle \approx \varepsilon \langle n | \Sigma^+ \pi^- \rangle$. But by our assumption (d), Section **1**, we have also that

$$\text{(33)}\qquad \langle n | \Sigma^+ \pi^- \rangle \approx \langle \Sigma^+ | n\pi^+ \rangle\,, \quad \text{so} \quad \langle \Upsilon^0 | p\pi^- \rangle \approx \varepsilon \langle \Sigma^+ | n\pi^+ \rangle\,.$$

H_s and $H_w(js)$ are also invariant for

$$\text{(34)}\qquad \begin{cases} N_1 \rightarrow -i\tau_2 N_3\,, \quad N_2 \rightarrow -i\tau_2 N_4\,, \quad \pi^\pm \rightarrow -\varepsilon \pi^\pm\,, \\ N_3 \rightarrow \;\; i\tau_2 N_1\,, \quad N_4 \rightarrow \;\; i\tau_2 N_2\,, \quad \pi^0 \rightarrow -\varepsilon \pi^0\,, \end{cases}$$

again if the permuted doublets are mass degenerate. Equ. (34) yields

$$\text{(35)}\qquad \langle Z^0 | p\pi^- \rangle \approx \varepsilon \langle n | \Sigma^- \pi^+ \rangle \approx \varepsilon \langle \Sigma^- | n\pi^- \rangle$$

by the same argument which led to equ. (33). From equs. (27), (33) and (35) we obtain

$$\varepsilon\langle \Lambda | p\pi^- \rangle \sqrt{2} \approx \langle \Sigma^+ | n\pi^+ \rangle + \langle \Sigma^- | n\pi^- \rangle \,, \tag{36}$$

which is the result stated in equ. (2). The ε in equ. (36) clearly has no physical significance.

Remarks.

1) The invariance under the transformations (32), (34) are the « sufficient symmetries » mentioned at the beginning of this section. For the present purposes these transformations embody the essence of $G^{\pm}$-invariance as applied to strong and weak interactions simultaneously. The commitment to the form (28) of the strong interactions is to some extent of secondary importance.

2) From the present derivation it follows that the less restrictive doublet approximation ([8]) is not sufficient ([22]) to obtain equ. (36).

3) What does it mean when we require for example that s is an S-violating T-spinor and with a G-symmetric structure? If we merely imply by the latter that s should be expressible in terms of mass degenerate doublets, there remains an ambiguity in the definition of s. Indeed, also

$$s^1 = \begin{pmatrix} s^1_- \\ s^1_0 \end{pmatrix} = \begin{pmatrix} \bar{N}_3\tau_3 N_1 - \bar{N}_4\tau_3 N_2 + \bar{N}_2\tau_- N_1\sqrt{2} + \bar{N}_4\tau_- N_3\sqrt{2} \\ \bar{N}_2\tau_3 N_1 + \bar{N}_4\tau_3 N_3 - \bar{N}_3\tau_+ N_1\sqrt{2} + \bar{N}_4\tau_+ N_2\sqrt{2} \end{pmatrix} \tag{37}$$

has all the desired properties. If we couple s^1 to j, however, there is no invariance for the transformations (32) and (34) and hence ([22]) there is no $\Sigma\Lambda$-relation (2) (*).

4) One easily verifies that equs. (32), (34) are also sufficient to derive equ. (6), for either G^+- or G^--symmetry.

In order to derive equ. (13) we observe that $H^{\mathrm{ch}}_w(js)$ cannot give rise of Σ^+-decay. (In the present $G^{\pm}$-approximation this is even true to all orders in the weak and strong interactions). Hence for the study of the Σ^+-modes it is legitimate to consider the mutilated interaction

$$H^1 = H_s + H^0_w(js) \tag{38}$$

(*) *Note added in proof.* - It can be shown that eq. (37) combined with a modified form of j does lead to eq. (2). A complete discussion of such further possibilities will be given in a sequel to this paper.

and to study its invariance properties. As long as we confine our attention to Σ^--channels it is admissible to operate with symmetry arguments which are violated by the dropped terms $H_w^{\mathrm{ch}}(js)$.

H^1 is invariant under equs. (32) and (34) and also under

$$(39)\qquad \begin{cases} N_1 \to i\tau_2 N_2\,, & N_3 \to i\tau_2 N_4\,, & \pi^{\pm} \to -\varepsilon\pi^{\mp}\,, \\ N_2 \to -i\tau_2 N_1\,, & N_4 \to -i\tau_2 N_3\,, & \pi^0 \to -\varepsilon\,\pi^0\,. \end{cases}$$

Oberve that $H_w^{\mathrm{ch}}(js)$ does not share this last property. From equ. (39) we immediately obtain equ. (13) by the same argument used in deriving equ. (33).

Remarks.

1) Of course $H_w^{\mathrm{ch}}(js)$ will contribute to the Σ^+-modes as soon as we allow G-violating strong interactions to participate. But these interactions will also modify equ. (36). Thus we repeat our main point: if one accepts or rejects equ. (36) one must do likewise with equ. (13).

2) If one applies equ. (39) to the $\Sigma^+ \to \mathrm{p}\pi^0$ mode, one does not obtain a condition which says that the amplitude vanishes if G^+-symmetry holds true. Nor is it possible to give an argument that implies $\langle \Sigma^- | n\pi^- \rangle \approx 0$ for either the G^+- or the G^--case ([22]).

3) The js-coupling also then satisfies $\Delta T = \frac{1}{2}$ if ([28]) one decomposes the currents j and s into the $(N_1, \Lambda, \Sigma, N_4)$-representation by means of equ. (27) and there upon gives all particles their real mass.

3. – The first possibility. G^--symmetry and js-coupling.

We first reiterate the conclusions to which we have come for this case: the $\Sigma^+ \to \mathrm{n}\pi^+$ mode presents no inner inconsistency with the simultaneous validity of equs. (2) and (3). The nearly parity-pure qualities of the modes $\Sigma^{\pm} \to \mathrm{n}\pi^{\pm}$ do not follow (approximately) from general argument, one would need more detailed dynamics to see if these properties could be understood in this version of the theory. As to the leptonic decays, the arguments of TREIMAN ([5]) summarized in equs. (7)–(10) also hold true here. The remaining question then is whether there exist other arguments for or against G^--symmetry.

([28]) See ref. ([5]); I am indebted to Professor TREIMAN for stressing the importance of this point in private discussions.

The relative sign of the $\Sigma\Sigma\pi$ as compared to the π-nucleon interaction which is at stake here ([29]) has of course important consequences as far as the Σ-nucleon interaction is concerned. It has been suggested ([30]) on the basis of K^--absorption experiments that the Σ-nucleon potential is attractive and has a depth $(30 \div 40)$ MeV. In this connection it has been pointed out by DALITZ ([31]) that this is difficult to understand if $\varepsilon = -1$, and that $\varepsilon = +1$ seems indicated, if G-symmetry is contemplated at all. On the other hand it has been noted ([32]) that K^--absorption data may not be very suited to test explicitly the validity of G-symmetries. For the present it seems therefore to be rather an open question whether evidence for or against $\varepsilon = -1$ exists.

From a theoretical point of view there exists a distinction between G^+- and G^--symmetry where the construction of approximately conserved S-violating currents is concerned. This can be seen from the recent work of GÜRSEY ([33]) which is based on an idea of TIOMNO ([34]) who uses G^--symmetry.

Finally we recall that this scheme fits in with the possibility of a universal interaction of the type (16) and that it is therefore compatible with the existence of a boson-quartet which carries all weak interactions ([14]).

4. – jt-coupling.

We introduce the quantities $\boldsymbol{t}_{ik}$ defined by

$$\boldsymbol{t}_{ik} = \bar{N}_i \boldsymbol{\tau} N_k \,; \qquad i, k = 1, \ldots, 4. \tag{40}$$

For the purposes of this section the space time structure of $\boldsymbol{t}_{ik}$ will not be relevant. We ask for such hermitian couplings $\boldsymbol{t}_{ik} \cdot \boldsymbol{t}_{jl}$ which satisfy $|\Delta S| = 1$. These are

$$\left.\begin{aligned} &\boldsymbol{t}_{ii} \cdot (\boldsymbol{t}_{12} + \boldsymbol{t}_{21}) \,, \\ &\boldsymbol{t}_{ii} \cdot (\boldsymbol{t}_{34} + \boldsymbol{t}_{43}) \,, \end{aligned}\right\} \quad i = 1, \ldots, 4, \tag{41}$$

([29]) In the present line of reasoning, the sign of the $\Sigma\Sigma\pi$ relative to the $\Sigma\Lambda\pi$-interaction is supposed to be positive in any case, as required by either kind of G-symmetry

([30]) F. C. GILBERT, C. E. VIOLET and R. S. WHITE: *Phys. Rev.*, **107**, 228 (1957).

([31]) R. DALITZ: *Proc. Intern. High Energy Conference* (CERN, 1958), p. 197. See also H. WEIZNER: same proceedings, p. 329.

([32]) A. SALAM: *Rep. of the Kiev Conf. on High Energy Physics* (1959).

([33]) F. GURSEY: *On the structure and parity of weak interaction currents*, preprint, *Phys. Rev.*, in course of publication.

([34]) J. TIOMNO: *Nuovo Cimento*, **6**, 69 (1957). The essential role of G^--symmetry in this work was pointed out to me by Dr. A. SALAM. See also R. BEHRENDS: *Nuovo Cimento*, **11**, 424 (1959).

and also (h. c. = hermitian conjugate)

$$\begin{cases} \boldsymbol{t}_{24}\cdot\boldsymbol{t}_{32} + \text{h. c.} \\ \boldsymbol{t}_{13}\cdot\boldsymbol{t}_{32} + \text{h. c.} \\ \boldsymbol{t}_{24}\cdot\boldsymbol{t}_{41} + \text{h. c.} \\ \boldsymbol{t}_{13}\cdot\boldsymbol{t}_{41} + \text{h. c.} \end{cases} \tag{42}$$

One easily verifies that the combination of interactions H_s of equ. (28) and any linear aggregate of the coupling (41), (42) satisfies to any order

$$\langle \Sigma^- | n\pi^- \rangle \approx 0 . \tag{43}$$

Equ. (43) is valid irrespective of whether we use G^+- or G^--invariance.

The interactions (41) and (42) differ in that every member of the second group contains currents $\boldsymbol{t}_{ik}$ which have components characterized by $\Delta Q = \pm 2$ while the couplings of equ. (41) have $|\Delta Q| \leqslant 1$. Currents with $|\Delta Q| = 2$ components lead only to unwanted complications, so we discard the interactions (42). As we are working toward G-symmetry we now consider jt couplings of the type

$$\begin{cases} \boldsymbol{j}\cdot(\boldsymbol{t}_{12} - \boldsymbol{t}_{21}) , \\ \boldsymbol{j}\cdot(\boldsymbol{t}_{34} + \boldsymbol{t}_{43}) , \\ \boldsymbol{j} = \sum_{i=1}^{4} \boldsymbol{t}_{ii} , \end{cases} \tag{44}$$

where $\boldsymbol{j}$ is therefore the same isotopic structure as defined in equ. (30). Clearly these couplings are of the $\Delta T = \frac{1}{2}$ type. Moreover, these interactions also then satisfy $\Delta T = \frac{1}{2}$ if one decomposes all currents into the $(N_1, \Lambda, \Sigma, N_2)$-representation by means of equ. (27) and thereupon gives all particles their real mass.

With an eye on the relations (2) and (6) we ask for the behaviour of the interactions (44) under the transformations (32) and (34). Each coupling is clearly invariant under the operation (32). To get also invariance under (34) we need both couplings with equal weight. Thus we now introduce the weak interaction $H_w(jt)$

$$\begin{cases} H_w(jt) = \boldsymbol{j}\cdot\boldsymbol{t} , \\ \boldsymbol{t} = \dfrac{1}{\sqrt{2}} (\boldsymbol{t}_{12} + \boldsymbol{t}_{21} + \boldsymbol{t}_{34} + \boldsymbol{t}_{43}) . \end{cases} \tag{45}$$

The factor $2^{-\frac{1}{2}}$ in the definition of t is convenient for future purposes. It follows now that any linear combination of H_s, $H_w(js)$ and $H_w(jt)$ gives rise to equs. (2) and (6).

Furthermore one easily verifies that $H_w(jt)$ is not invariant under the operation (39). Thus for jt-coupling there is no analog to a relation like equ. (13).

Finally we observe that the t-current is a mixture of $T=\frac{1}{2}$ and $\frac{3}{2}$. For example $t_{21}^{-}=2^{-\frac{1}{2}}(\bar{\Lambda}-\bar{\Sigma}^0)p$. Here we used the definitions

$$t^{\pm}=\frac{1}{\sqrt{2}}(t_1\pm it_2)\,,\qquad t^0=t_3\,, \tag{46}$$

in analogy with equ. (30). It also follows that t contains terms of both the types $\Delta S/\Delta Q=\pm 1$.

5. – The second possibility. G^+-symmetry and (js), (jt)-coupling.

a) *Non-leptonic decays.* – Consider the combined strong and weak interaction

$$H=H_s+\text{const}\,[H_w(js)+\lambda H_w(jt)]\,, \tag{47}$$

where the terms on the right hand side are defined by equs. (28), (29) and (45) λ is the relative weight of jt- and js-coupling. Until further notice all results will be independent of the value of λ, as long as $\lambda\neq 0$. We stipulate expressly that in all that follows the parameter ε which occurs in equ. (28) shall be $=+1$.

Let us now summarize what we can say about the Σ-modes if we adopt equ. (47).

1) $\Sigma^+\to \mathrm{n}\pi^+$ proceeds and only proceeds via jt-coupling, see equ. (11).

2) $\Sigma^+\to \mathrm{p}\pi^0$ proceeds via jt-coupling and via H_w^0-(js), see Sect. **2** and **4**.

3) $\Sigma^-\to \mathrm{n}\pi^-$ proceeds and only proceeds via js-coupling see equ. (43).

We repeat that these and later results are valid in the approximation in which non-G^+-symmetric strong interactions are ignored. Thus we conclude, as already stated in Section **1**:

If $H_w(js)$ and $H_w(jt)$ are separately P-conserving then the modes $\Sigma^{\pm}\to \mathrm{n}\pi^{\pm}$ are each parity conserving. If furthermore $H_w(js)+\lambda H_w(jt)$ is P-violating then $\Sigma^+\to \mathrm{p}\pi^0$ is P-violating. The same is then also true for the Λ- and the Ξ-modes in virtue of equs. (2) and (6).

In order to achieve this structure of the weak non-leptonic interaction we must now finally specify in more detail the space-time properties of the currents. Here we shall first of all assume, as seems most natural, that only V- and A-currents occur. We leave aside the question of the possible presence and of the magnitude of induced currents. Thus we look upon equ. (47) as an (approximate) true rather than an effective interaction. There are two essentially distinct ways in which we can reach our purpose.

A) The same pure j (either A or V) occurs in js and in jt; while either $s = A$, $t = V$ or $s = V$, $t = A$.

Example. $j = V$, $s = A$, $t = V$. js contains terms like $\overline{\Sigma^-}\gamma_\lambda\gamma_5 n \cdot \overline{p}\gamma_\lambda n$ and $\overline{\Sigma^+}\gamma_\lambda\gamma_5 p \cdot [\overline{p}\gamma_\lambda p - \overline{n}\gamma_\lambda n]$; jt contains terms like $\overline{\Sigma^+}\gamma_\lambda n \cdot \overline{n}\gamma_\lambda p$. The first one is representative for Σ^--decay only; the second one for $\Sigma^+ \to p\pi^0$ only; the third one for both Σ^+-modes.

Note. Here and in the following it is only a question of words whether and by which interaction we define the parity of a Σ relative to nucleons. What only matters is whether P-interference does or does not occur.

B) s and t are either both A or both V, while the j's occurring in js and jt are pure and opposite. This still leaves four possibilities which, in obvious shorthand can be characterized as follows,

$$\left\{\begin{array}{lll} a) & j^A s^V, & j^V t^V, \\ b) & j^V s^V, & j^A t^V, \\ c) & j^A s^A, & j^V t^A, \\ d) & j^V s^A, & j^A t^A. \end{array}\right. \tag{48}$$

Remark. There are eight further possibilities of choosing individually pure currents, each one either V or A. All these cases are of no interest as they imply P-conservation for all three Σ-modes.

We have already argued in Section **1** that case *B*) seems more attractive than case *A*). And from hyperon β-decay, to be discussed shortly, we have tentatively stated that the cases *a*) and *b*) with both s and t pure V perhaps seem indicated. Also we have mentioned in Section **1** what observations may distinguish between *a*) and *b*).

Let us next write down the weak interaction part of equ. (47) assuming furthermore that λ occurring in equ. (47) is given by

$$\lambda = +1 \ (?) \,. \tag{49}$$

This yields

$$(50) \qquad \text{case } a) \quad \bar{\Sigma}^- \gamma_\lambda n \cdot j_\lambda^{A+} + \bar{\Sigma}^+ \gamma_\lambda n \cdot j_\lambda^{V-} - \frac{1}{\sqrt{2}} \bar{\Sigma}^+ \gamma_\lambda p [j_\lambda^A - j_\lambda^V]^0 + \\ + \frac{1}{\sqrt{2}} \left[\bar{\Lambda} \gamma_\lambda p (j_\lambda^A + j_\lambda^V)^+ - \frac{1}{\sqrt{2}} \bar{\Lambda} \gamma_\lambda n (j^A + j^V)^0 \right] + \dots ,$$

where for the purpose of illustration we have written out the terms which couple Σ^- and Λ to nucleons. The other cases mentioned in equ. (48) follow in a similar manner ([35]). The important and open question is: would it be feasible to show that equ. (50) and the strong interactions combined lead to an effective decay interaction of the form

$$(51) \qquad \Sigma^- \gamma_\lambda n \cdot \frac{\partial \pi^-}{\partial x_\lambda} + r \, \bar{\Sigma}^+ \gamma_\lambda \gamma_5 n \cdot \frac{\partial \pi^+}{\partial x_\lambda} + \dots ,$$

where r is a number $\approx 1, 2$? If so we would have a reasonable agreement with equ. (3), as shown by BLUDMAN ([3]). (The present cases *a*) and *b*) find their counterpart in Bludman's solutions 1 and 2)). In addition one would like to have ([3]) an effective $\Lambda \to \mathrm{p}\pi^-$ decay coupling of the form $\bar{\Lambda} \gamma_\lambda (1 + r\gamma_5) p \cdot \partial \pi^- / \partial x_\lambda$ in order to get the correct sign and order of magnitude of the proton helicity in Λ-decay ([7]). In the present instance these questions are tied to the value of λ. I do not know whether the value (49) of λ which leads to the suggestive form (50), or any other value, will lead to the desired result.

Remark. Equ. (50) illustrates that the S-violating currents remain pure V (or A) if the hyperons are given their real masses. But of course some P-violation in $\Sigma^+ \to \mathrm{n}\pi^+$ is introduced by so doing. If the general ideas outlined here were to prove correct, it would become an important problem to understand why this amount of P-violation is small. A further potential source of such P-violation will be encountered in the next subsection.

b) *Leptonic decays.* – First we discuss a direct consequence of the assumptions *f*) and *g*) stated in Section **1**. The S-violating currents s and t are pure and of the same type; and these same currents intervene in the S-violating leptonic decays. Thus the S-violating current coupled to the usual lepton current J is written ans const $(s + \varrho t)$, where ϱ is again a relative t to s

([35]) To get case *b*), interchange the superscripts V and A in eq. (50). To get cases *c*) and *d*) from *a*) and *b*) respectively, replace γ_λ by $\gamma_\lambda \gamma_5$.

weight which need for the moment not further be specified. (Nor need it necessarily be equal to λ introduced in equ. (47).)

Let us now consider the K^+_{l2} and K^+_{l3} modes, where l stands for either μ or e. Here for the first time in this work we need to consider also strong K-interactions which, in whole or in part, must violate the presently assumed G-symmetry. However, all we need to know for the argument is that the strong K-interactions conserve parity.

The amplitudes for these two K-modes can be written as follows:

$$(52)\qquad \begin{cases} K^+_{l2}: & \langle 0|s^- + \varrho t^-|K^+\rangle\, \bar{u}_l \gamma_\lambda (1+\gamma_5) u_\nu\,, \\ K^+_{l3}: & \langle \pi^0|s^- + \varrho^-|K^+\rangle\, \bar{u}_l \gamma_\lambda (1+\gamma_5) u_\nu\,. \end{cases}$$

Suppose now that the Ξ-nucleon parity $p(\Xi)$ were even. As is well known, this parity is only defined with respect to the K-part of the strong interactions. This has of course entitled us to bypass in the G-approximation (as in equ. (47)) the question of $p(\Xi)$ altogether. For even parity we now have a paradox. Indeed, in this case the parity of K is the same with respect to all allowed baryon pairs; but then if the S-violating current is pure, K has also a well defined parity relative to π. Hence one or the other of the reactions (52) would be strictly forbidden in contradiction with experiment.

No such paradox arises if $p(\Xi)$ is odd. Let us denote by Y either a Σ- or the Λ-particle. (Note that the $\Sigma\Lambda$-parity is supposed to be even in any case as we are assuming G-symmetry). Then for odd $p(\Xi)$ the situation can be summarized as follows.

If K^+ is scalar (pseudoscalar) relative to $\bar{Y}N_1$ and if $s+\varrho t$ is of the $V(A)$ type, then K^+_{l2} goes via $\bar{Y}N_1$-interactions and K^+_{l3} via $\bar{N}_1 Y$-interactions. If K^+ is scalar (pseudoscalar) relative to $\bar{Y}N_1$ and $s+\varrho t$ is of the $A(V)$ type then K^+_{l2} goes via $\bar{N}_1 Y$ and K^+_{l3} via $\bar{Y}N_1$-interactions.

Example. K^+ pseudoscalar relative to $\bar{Y}N_1$, $s+\varrho t$ pure V. We then have strong coupling terms of the type $\bar{p}\gamma_5 \Lambda K^+$ and $\bar{\Lambda}\Xi^- K^+$. The simplest graph for K^+_{l2} is via a cascade-Λ loop, for K^+_{l3} we go via a Λ-nucleon loop.

The situation for K^0_{l3} is like the one for K^+_{l3}. Note that this in itself does not fix ([36]) the relative parity of K^+ and K^0.

Because of the importance of this parity question it may be well to state concisely what we actually mean by calling $s+\varrho t$ pure also in the presence

([36]) If this parity is even the $\bar{Y}N$-interactions play the same role for K^+ as for K^0; likewise for $\bar{N}_1 Y$. If this parity is odd, one must interchange the roles of nucleon and cascade coupling in going from K^+ to K^0. Note that, at least in the G-approximation, this alternative does not influence the K^0_{e3}/K^+_{e3} rates.

of K-interactions. Let us assume that s and t are of the γ_λ-type and write out $s^- + \varrho t^-$ in full:

$$s^- + \varrho t^- = T_\lambda + U_\lambda ,$$

$$T_\lambda = \bar{Z}^0 \gamma_\lambda p + \bar{\Sigma}^- \gamma_\lambda n + \varrho(\bar{Y}^0 \gamma_\lambda p + \bar{n} \gamma_\lambda \Sigma^+) , \tag{53}$$

$$U_\lambda = \bar{\Xi}^0 \gamma_\lambda \Sigma^+ + \bar{\Xi}^- \gamma_\lambda Y_0 + \varrho(\bar{\Sigma}^- \gamma_\lambda \Xi^0 + \bar{\Xi}^- \gamma_\lambda Z^0) . \tag{54}$$

We distinguish two cases:

1) Absence of K-coupling. The behaviour under space reflections of Ξ relative to nucleon is not defined by any interaction, weak or strong. We may call it plus by convention, then T_λ and U_λ are both vectors. Had we chosen the convention minus then one of the two quantities T_λ, U_λ would be a vector the other a pseudovector. This is a question of words; it would in no way affect the non-leptonic decays. In this case it makes no difference whether we call a current « pure V » or « pure γ_λ ». (Likewise for A and $\gamma_\lambda\gamma_5$).

2) Presence of strong K-coupling. Now p(Ξ) has physical meaning. Odd p(Ξ) means that $s + \varrho t$ is a parity mixture of T_λ and U_λ one of which is a vector the other a pseudovector. What subsists however is the idea of a pure γ_λ (or pure $\gamma_\lambda\gamma_5$) structure of the (unrenormalized) S-violating current.

Let us now return to the question of the non-leptonic modes. Under the conditions of G-symmetry we obtained P-conserving modes because the S-violating currents are truly V-currents (or A-currents) in that approximation. Now we see that the necessity of odd p(Ξ) implies that virtual K-effects will lead to parity mixing of these γ_λ (or $\gamma_\lambda\gamma_5$) currents. This only emphasizes once more the fact that the G-approximation can only then possibly be of practical use if virtual K-effects do not strongly distort the non-leptonic decay amplitudes.

We have already stated that the equs. (22)–(24) are independent of the value of ϱ (of course equs. (9) and (10) would be true for $\varrho = 0$). As an example we prove equ. (22). The leptonic S-violating decay interaction is

$$H_\beta(\Delta S) = (s^- + \varrho t^-) J^+ + (s^+ + \varrho t^+) J^- . \tag{54a}$$

To the first order in the weak interactions we may write every amplitude as due additively to the s and the t-current, for example

$$\langle \Lambda | p e^- \bar{\nu} \rangle = \langle \Lambda | p e^- \bar{\nu} \rangle_s + \langle \Lambda | p e^- \cdot \bar{\nu} \rangle_t ,$$

and we may treat the s- and t-parts separately. Clearly

$$\langle Y^0 | ne^-\bar{\nu}\rangle_s \approx 0\,, \tag{55}$$

$$\langle Z^0 | pe^-\bar{\nu}\rangle_s \approx \langle \Sigma^- | ne^-\bar{\nu}\rangle_s\,, \tag{56}$$

where equ. (56) is a consequence of the invariance of the strong and Js-coupling under $N_\alpha \to i\tau_2 N_\alpha$, $\pi^\pm \to -\pi^\mp$, $\pi^0 \to -\pi^0$. Thus from equs. (27), (55), (56)

$$\langle \Sigma^- | ne^-\bar{\nu}\rangle_s \approx \sqrt{2}\langle \Lambda | pe^-\bar{\nu}\rangle_s\,, \tag{57}$$

which is in fact Treiman's relation ([5]) equ. (7). Also

$$\langle \Sigma^+ | ne^+\nu\rangle_s \approx 0\,. \tag{58}$$

Likewise

$$\langle Z^0 | ne^-\bar{\nu}\rangle_t \approx 0\,, \tag{59}$$

$$\langle \Sigma^+ | ne^+\nu\rangle_t \approx \langle Y^0 | pe^-\bar{\nu}\rangle_t\,, \tag{60}$$

where equ. (60) is a consequence of the invariance of the strong and Jt-coupling under $N_\alpha \to i\tau_2 N_\alpha$, $\pi^\pm \to -\pi^\mp$, $\pi^0 \to -\pi^0$ and $J^\pm \to -J^\mp$. From equs. (27) (59) and (60)

$$\langle \Sigma^+ | ne^+\nu\rangle_t \approx \sqrt{2}\langle \Lambda | pe^-\bar{\nu}\rangle_t\,, \tag{61}$$

while also

$$\langle \Sigma^- | ne^-\bar{\nu}\rangle_t \approx 0\,. \tag{62}$$

Equ. (22) now follows from equs. (57), (58), (61) and (62) and the quoted result is independent of ϱ.

There is presently too little to go on experimentally to do more than give the consequences of what seems a rather natural choice for ϱ, namely

$$\varrho = +1 \quad (?)\,. \tag{63}$$

In this case we see from equs. (53), (54), (54a) and (27) that

$$H_\beta(\Delta S;\ \varrho = 1) = (\sqrt{2}\,\bar{\Lambda}\gamma_\lambda p + \bar{\Sigma}^-\gamma_\lambda n)J_\lambda^+ + \bar{\Sigma}^+\gamma_\lambda n J_\lambda^- + \tag{64}$$
$$+ (\sqrt{2}\,\bar{\Xi}\gamma_\lambda \Lambda + \bar{\Xi}^0\gamma_\lambda \Sigma^+)J_\lambda^+ + \bar{\Xi}^0\gamma_\lambda \Sigma^- J_\lambda^- + \text{h. c.}$$

Implications of this form have been discussed in connection with equ. (25). Observe that the form (64) does not contain any Σ^0. Thus in lowest Born approximation we have for example that $\Xi^- \to \Sigma^0 + e^- - \bar{\nu}$ is suppressed.

c) *The second possibility. Summary.* – In conclusion we collect the main aspects of the dynamical scheme discussed in Section **5** *a*), *b*).

1) The theory is committed to G^+-symmetry. This implies in particular that the $\Sigma\Lambda$-parity should be even. Suggestions have been made concerning the determination of this parity ([37]). It implies also that the relative sign of the $\Sigma\Sigma\pi$- and π-nucleon interaction is plus. For comments on this point see Section **3**.

2) The near parity-pure features of the modes $\Sigma^+ \to n\pi^+$ are built into the dynamics of the theory, and in fact constitute its raison d'être. On this problem the first possibility has nothing to say ([22]). The approximate experimental relation (3) has not been derived, although it is not in any obvious conflict with the theory; on this point see the discussion concerning equs. (49), (50) and (51).

3) This theory is committed to (approximate) relations between up-down asymmetry parameters in $\Sigma^+ \to p\pi^0$, $\Lambda \to p\pi^-$; for Λ- and Ξ-decay see equs. (2) and (6*a*). In itself these relations, if verified, would not discriminate between the first and the second possibility.

4) The simplest possibility is that the S-violating currents which mediate non-leptonic decay are all of the pure V (*i.e.* γ_λ) or of the pure A (*i.e.* $\gamma_\lambda\gamma_5$) type. There is one alternative to this statement, see Section **5**-*a*), case A; however, the latter case seems less natural.

5) If we assume that the same S-violating currents mediate both the non-leptonic and the leptonic decays of K-particles and hyperons, it follows that the Ξ-nucleon parity is odd. Suggestions have been made concerning the determination of this parity ([11,38]).

6) Within this second possibility there are at least two causes for some P-violation to occur in $\Sigma^\pm \to n\pi^\pm$. (These reasons may or may not be interrelated). The first one arises when one gives the hyperons their real masses

([37]) G. Feinberg: *Phys. Rev.*, **109**, 1019 (1958); A. Pais and S. Treiman: *Phys. Rev.*, **109**, 1759 (1958); G. Feldman and T. Fulton: *Nucl. Phys.*, **8**, 106 (1958).

([38]) L. B. Okun', I. Ia. Pomeranchuk and I. M. Shmushkevich: *Žurn. Eksp. Teor. Fiz.*, **34**, 1246 (1958); translation *Journ. Exp. Theor. Phys.*, **7**, 862 (1958); S. Treiman: *Phys. Rev.*, **113**, 355 (1959); J. Szymanski: *Nuovo Cimento*, **11**, 730 (1959).

(Section 5-*a*)). The second one comes from vertex corrections due to K-couplings and the oddness of the Ξ-nucleon parity (Section 5 *b*)). It would be important to explain why these corrections are small. In this context more precise measurements of the up-down asymmetries for these two modes acquire special significance.

7) The apparent suppression of hyperon β-decay is tentatively linked to a particular choice of the *S*-violating current, namely that it is a *V* (*i.e.* γ_λ) current.

8) The β- and μ-decay of the Σ^+ and the negative electron and μ^--modes of K^0_{l3} are in principle allowed reactions; they could only be forbidden in an artificial way.

9) The most crucial tests of the theory may well come from details of hyperon β-decay, especially if the problem of the so-called induced terms could be handled ([25]).

10) The (*js*), (*jt*) coupling scheme is a special instance of parity clash. This last concept is incompatible with the idea that all weak interactions can be comprised in an (overall current)×(overall current) structure.

11) The theory is therefore incompatible with the idea ([14]) that all weak interactions are effectively carried by a quartet of vector bosons.

12) This theory is not incompatible with Pontecorvo's suggestion ([39]) that the neutrinos accompanying electrons need not be identical with those that accompany μ-mesons. However, there is no such obvious need for the existence of distinct neutrinos as is the case for the boson-quartet picture ([11]).

13) From the point of view of this theory there is no such question as the mysterious absence of neutral lepton currents .

* * *

I would like to express my gratitude to CERN for its kind hospitality and for financial support through funds of the Ford Foundation. I am indebted to Drs. B. d'Espagnat, J. Prentki and S. Treiman for discussions.

([39]) B. Pontecorvo: *Žurn. Èksp. Teor. Fiz.*, **37**, 1751 (1959); translation *Journ. Exp. Theor. Phys.*, **10**, 1236 (1960). See also reference ([14]).

RIASSUNTO (*)

L'na recente proposta dovuta a TREIMAN (simmetria globale ed accoppiamento *js* del decadimento non leptonico) costituisce il punto di partenza di questo lavoro. Si nota una incongruenza intrinseca della teoria: $\Sigma^+ \to n\pi^+$ è proibita nell'approssimazione adottata. Questo può significare che le ipotesi fondamentali di simmetria della teoria devono essere rifiutate. D'altra parte ci sono due possibilità di evitare l'incongruenza, se necessario, pur conservando nello stesso tempo le predizioni sul decadimento non leptonico. *Prima possibilità*: segno relativo ε della $\Sigma\Sigma\pi$ riferentesi alla interazione π-nucleone $= -1$. *Seconda possibilità*: $\varepsilon = +1$. L'interazione *js* è sostituita da una più generale (accoppiamento *js* e *jt*). Questo accoppiamento è soggetto a un conflitto di parità: *js* e *jt* separatamente conservano la P, ma assieme violano la P. Di conseguenza entrambe le reazioni $\Sigma^\pm \to n\pi^\pm$ conservano la P nella approssimazione della simmetria globale. Il conflitto si realizza nel modo più semplice se nella approssimazione della simmetria globale esiste un solo tipo di corrente che viola l'S: o V od A. Se si suppone che la stessa corrente che viola l'S si trovi sia nel decadimento leptonico, sia nel decadimento non leptonico, ne segue che la parità Ξ-nucleone deve essere dispari. Si mostra che il concetto di conflitto di parità è incompatibile con una struttura (corrente complessiva)$\times$(corrente complessiva) per la totalità delle interazioni deboli. È incompatibile con l'idea che un quartetto bosonico vettoriale porta tutte le interazioni deboli. Questa possibilità deve passare molte prove (riassunte alla fine) per sopravvivere. Nella sua forma più naturale permette che avvengano le reazioni $\Sigma^- \to n + e^+ + \nu$, $K^0 \to \pi^+ + e^- + \bar{\nu}$. Nella versione più semplice dei decadimenti leptonici la corrente carica leptonica stessa ha un ruolo analogo al potenziale quadrivettoriale in elettromagnetismo. Tutti gli aspetti dei decadimenti che non conserano S non vengono influenzati.

(*) *Traduzione a cura della Redazione.*

Quantum Excitation and Radiational Damping of Electron Oscillations in Cyclic Accelerators.

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(ricevuto il 21 Luglio 1960)

Elaboration of a full theory of electron motion in cyclic accelerators and storage systems requires experimental data about quantum excitation laws as well as about radiational damping of betatron and synchrotron oscillations of electrons.

The possibility of oscillation excitation under the influence of quantum fluctuations of radiation of electrons was first theoretically predicted by A. A. Sokolov and I. M. Ternov [1] and developed in the theory of the macroatom of these authors and D. Ivanenko [2]. The existence of radiational damping and its rather considerable influence on the electron motion in synchrotrons was first shown by A. A. Kolomensky and A. N. Lebedev [3]. However, up to now these important theoretical works have not found sufficient experimental verification. There exists only a single experimental investigation of M. Sands, treating quantum excitation of synchrotron oscillations [4]. But no study of quantum excitation and radiational damping of betatron oscillations was undertaken until recently. The present investigation was undertaken with the aim to fill these gaps.

The experiment was carried out on the S-60 synchrotron of race-track type in the Lebedev Physical Institute of the Academy of Sciences of the USSR with a maximal energy of accelerated electrons equal to 660 MeV and with an acceleration time of 0.6 s. A sufficiently high value of maximal energy and a comparatively long acceleration time offered good conditions for the investigation of quantum excitation phenomena as well as of radiational damping of oscillations.

Experimental data were obtained by fast photographic recording of the radiation emitted by accelerated electrons with subsequent photometrical investigation of the pictures. The radiation was led out from dough-nut by means of a mirror through the window in the flange of the radial sleeve. The number of electrons in the beam at the photograph was about 10^7. The speed of recording was about 500 pictures per s.

(1) A. A. Sokolov and I. M. Ternov: *Žurn. Éksp. Teor. Fiz.*, **25**, 698 (1953).
(2) A. A. Sokolov, D. Ivanenko and I. M. Ternov: *Dokl. Akad. Nauk SSSR*, **111**, 334 (1956).
(3) A. A. Kolomensky and A. N. Lebedev: *Dokl. Akad. Nauk SSSR*, **106**, 807 (1956).
(4) M. Sands: *Report at the Intern. Conf. on High-Energy Accelerators*, Geneva (Sept. 1959).

This allowed one to obtain during a single acceleration cycle some $(200 \div 250)$ useful pictures, which covered the energy interval from 100 to 660 MeV. In each series of pictures of a single acceleration cycle about 40 pictures were measured, separated by a time interval of 0.01 s. Photometric investigation was made with the aim to obtain some data about axial and radial magnitudes of the beam and about its changes during the process of acceleration. The density distribution in the image of the beam on each photograph characterizes the co-ordinate distribution of electrons in the cross section of the beam, averaged over the time of exposition, which equals $4 \cdot 10^{-4}$ s. The investigation of photographs consists in obtaining microphotograms of the beam, made along two mutually orthogonal directions, which pass across the point of maximum density. One of them was taken parallel to the magnetic field, the other parallel to the accelerator radius (see Fig. 1). These microphotograms allow to obtain the absolute values of the width of axial and radial distribution of electrons in the beam at the levels 0.2; 0.3; 0.4; 0.5; 0.6 from the maximum of the corresponding distribution.

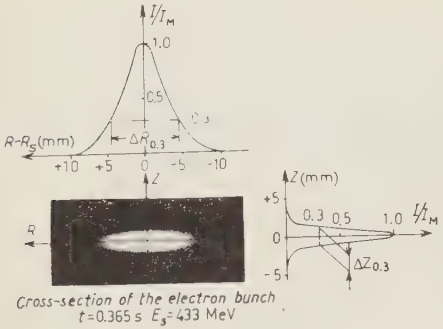

Fig. 1. – The photograph of the beam cross section at $t = 0.365$ s after the injection moment ($E_s = 433$ MeV) and its radial and axial radiation intensity distribution. Here I = radiation intensity; I_m = maximum radiation intensity, which corresponds to the point of maximum optical density in the image of the beam.

In comparing the experimental and theoretical data it was assumed, that the law of change of the beam width must in first approximation correspond to the change of the oscillation amplitude. We have used the formulae of paper [5] and have written for the amplitude of axial oscillations

$$(1) \qquad a_z = \left| a_{z_0}^2 \frac{E_{s_0}}{E_s} \exp\left[-\int_{t_0}^{t} \frac{W_s}{E_s} \mathrm{d}t' \right] + \frac{13}{24\sqrt{3}} \frac{r_0 A c}{R_s n \gamma} \int_0^t \exp\left[-\int_{t''}^{t} \frac{W_s}{E_s} \mathrm{d}t' \right] \gamma^4 \mathrm{d}t'' \right|^{\frac{1}{2}},$$

and for the total amplitude of radial oscillations

$$(2) \quad a_r = \left\{ \left| a_{r\dot{p}_0} \left(\frac{E_{s_0}}{E_s} \right)^{\frac{1}{2}} \exp\left[-\frac{1}{2} \frac{n}{1-n} \int_{t_0}^{t} \frac{W_s}{E_s} \mathrm{d}t' \right] + a_{rs_0} \left(\frac{V}{V_0} \right)^{\frac{1}{4}} \left(\frac{E_{s_0}}{E_s} \right)^{\frac{3}{4}} \exp\left[-\frac{1}{2} \frac{3-4n}{1-n} \int_{t_0}^{t} \frac{W_s}{E_s} \mathrm{d}t \right. \right. \right.$$

$$+ \frac{55}{24\sqrt{3}} \frac{r_0 A c}{R_s (1-n)^2 \gamma} \int_0^t \exp\left[-\frac{n}{1-n} \int_{t'}^{t} \frac{W_s}{E_s} \mathrm{d}t' \right] \gamma^6 \mathrm{d}t'' +$$

$$\left. \left. \left. + \frac{55}{24\sqrt{3}} \frac{r_0 A c V^{\frac{1}{2}}}{R_s (1-n)^2 \gamma^{\frac{3}{2}}} \int_0^t \exp\left[-\frac{3-4n}{1-n} \int_{t''}^{t} \frac{W_s}{E_s} \mathrm{d}t' \right] \gamma^{6\frac{1}{2}} V^{-\frac{1}{2}} \mathrm{d}t'' \right\}^{\frac{1}{2}} \right.$$

[5] A. A. KOLOMENSKY and A. N. LEBEDEV: *Atomnaya Energ.*, **4**, 31 (1957).

Here a_z, a_{rb} and a_{rs} mean amplitude of axial betatron, radial betatron and radial synchrotron oscillations respectively; E_s: equilibrium electron energy; W_s: energy flux radiated by an electron per s; r_0: classical radius of the electron; λ: Compton wave length; R_s: radius of the equilibrium orbit; n: field index; γ: ratio of the total energy of the electron to its rest energy; V: value of the RF voltage on the resonator. The index « 0 » means that the corresponding value is taken at the moment t_0. In our case $t_0 = 0.147$ s.

The comparison of experimental and theoretical data was made for the time interval from re-capture to the end of the acceleration cycle. Re capture corresponds to the moment of t, when one of the two resonators of the synchrotron is switched off while the other one is entering into action. In the cycles of acceleration, for which the data were used in our paper, re-capturing occured at 0.147 s after injection at the electron energy 184 MeV.

Results for the axial and radial widths of the bunch are given in Fig. 2 and Fig. 3 respectively as the curves, showing the width as function of the time passed

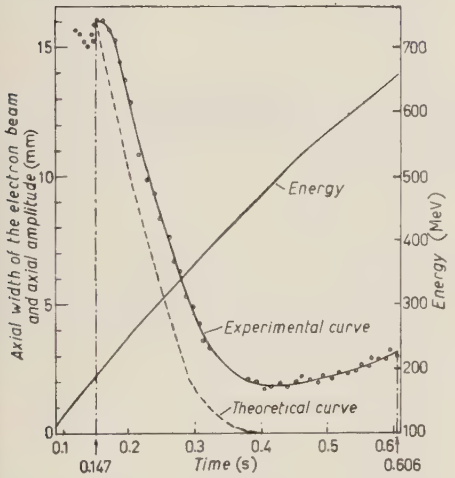

Fig. 2. – Plot of the axial beam width, axial oscillation amplitude and equilibrium electron energy *vs.* time from injection moment.

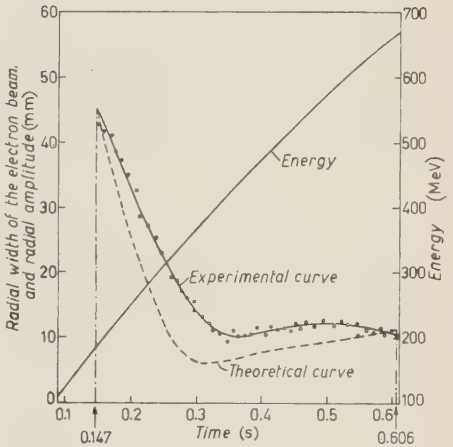

Fig. 3. – Plot of the radial beam width, total radial oscillation amplitude and equilibrium electron energy *vs.* time from injection moment.

from the injection moment. In the same plots there are shown the theoretical curves of the change of doubled amplitude values of the axial and radial oscillations, calculated by means of (1) and (2). Also the curve of the change of the equilibrium energy of the electrons is given on these figures. We have compared the theoretical curve with the experimental width change at the level 0.3. This level was chosen, because this width in photometrical investigation is measured in the most definite way. In constructing the theoretical curve for the change of amplitude we have taken for $2a_{z_0}$ the value of axial width of the beam at the moment of re-capture equal to 16 mm.

Initial amplitudes of radial betatron and synchrotron oscillations were determined in the following way. As it is shown by experiment the radial beam width before re-capture is equal to 21 mm and seems to be due to betatron oscillations only. At the time of re-capture the radial dimensions of the beam are sharply increased

and this width becomes equal to 45.4 mm, which is due to the starting of synchrotron oscillations.

Due to these circumstances we have taken for $2a_{r\dot{b}_0}$ the value of the width of the beam before re-capture equal to 21 mm and for $2a_{rs_0}$ we have taken the difference of width before and after re-capture equal to 24.4 mm.

Analysing the experimental data and comparing them with the theoretical curves we can draw the following conclusions.

1) The theory, which takes into account radiational damping, as was first developed by A. A. KOLOMENSKY and A. N. LEBEDEV, proves to be in satisfactory agreement with experiment, although there is yet no exact agreement.

2) In the energy region over 400 MeV in the Academy of Sciences synchrotron S-60 one has a pronounced excitation of radial betatron and synchrotron oscillations due to radiation quantum fluctuations first predicted by A. A. SOKOLOV and I. M. TERNOV and developed in the theory of the macroatom of these authors and D. IVANENKO.

3) One observes undamped axial oscillations whose presence was not predicted by the theory.

4) In the energy region over 550 MeV in the machine S-60 one observes some increase of the axial dimensions and a small decrease of the radial dimensions of the beam which also were not predicted theoretically.

* * *

The authors acknowledge the friendly assistance of M. S. RABINOVITCH, V. A. PETUKHOV, V. E. PISAREV in the work on the synchrotron S-60 and the instructive discussions on theoretical aspects of the problem with A. A. SOKOLOV, D. IVANENKO, I. M. TERNOV, A. A. KOLOMENSKY and A. N. LEBEDEV.

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

The Flux of Primary Cosmic Ray α-Particles Over Southern England.

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(ricevuto il 26 Agosto 1960)

The flux of α-particles in the primary cosmic radiation has been measured in a stack of nuclear emulsions exposed on the 20th of November 1958 over Southern England at a geomagnetic latitude of 55° N. The stack consisted of twenty 600 μm Ilford G-5 stripped nuclear emulsions measuring 10 cm × 10 cm and was exposed for (5.00 ± 0.25) h at a mean altitude of 86 500 ft (20.7 g/cm²), allowance being made for the time spent in ascent and descent.

The tracks produced by α-particles entering the stack were detected by scanning the emulsions along a line 1 cm below the top edge. Those tracks which satisfied the following selection criteria were recorded:

(i) a projected length of more than 6 mm per emulsion (10 mm in a subsidiary experiment);

(ii) a grain density greater than about three times minimum;

(iii) a zenith angle of less than 60°.

In a scanning length of 35 cm in the main experiment and 10 cm in the subsidiary experiment 434 tracks were noted. Measurements were made of multiple scattering and grain density and clear resolution was obtained between the 110 α-particles and 312 singly charged particles. In addition 12 heavier nuclei were observed.

As a check on the efficiency of detection of α-particles, 10 cm of the emulsion were rescanned and this revealed no new tracks. An integral length distribution of all the tracks showed no evidence for the missing of short tracks in the main experiment. To allow for possible scanning losses in the subsidiary experiment only those tracks of length exceeding 11 mms were used. The measurement of the depth in the emulsion at which each track crossed the scan line indicated a deficiency of tracks near the glass and consequently in calculating the flux the bottom tenth of the emulsion and the five α-particles in it, were ignored.

None of the α-particles was found to originate from the breakup of heavier nuclei in the overlying emulsion. The flux of α-particles at the scan line was (77 ± 8) α-particles/m²·sr·s. To extrapolate this value to the top of the atmosphere corrections were made for the absorption of α-particles in the emulsion and for both the absorption and creation

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of α-particles in the atmosphere and packing materials (0.7 g/cm²) above the stack. A simplified version of the diffusion equation due to KAPLON *et al.* [1] which neglects the influence of consecutive fragmentation was used [2]. In applying this equation it was assumed that the fluxes at the top of the atmosphere of α-particles, L-nuclei, M-nuclei and H-nuclei were in the ratio 100 : 2.2 : : 6.4 :2.9 [3]. The values taken for the absorption mean free paths in air were: $\lambda_\alpha=48$, $\lambda_L=36$, $\lambda_M=32$ and $\lambda_H=27$ g/cm² whilst for emulsion $\lambda_\alpha=75$ g/cm². Fragmentation parameters in air were: $P_{L\alpha}=$ $=0.81$, $P_{M\alpha}=0.93$, $P_{H\alpha}=1.30$, $P_{LL}=0.13$, $P_{MM}=0.16$ and $P_{HH}=0.31$ [3]. The flux of α-particles at the top of the atmosphere was found to be (121 ± 15) α-particles/m²·sr·s. The error quoted here includes a possible error of 5% in the time of flight and a possible error of 10% in the value for λ_α for air, but neglects any error in the assumed thickness of the emulsion.

This result may be compared with that of (162 ± 17) α-particles/m²·sr·s of energy greater than $(0.65\pm.05)$ GeV per nucleon obtained by WADDINGTON over Southern England at a time of sunspot minimum, 9th July 1954 [4,2]. This cut-off energy, taken in conjunction with the energy spectrum determined during the latter part of 1957 by FREIER *et al.* [5] would predict a flux value at a time of sunspot maximum of (107 ± 19) α-particles/m²·sr·s. This is in good agreement with the value obtained here, showing that the flux at this time was still depressed to the solar maximum value.

The author wishes to thank Professor C. F. POWELL for the hospitality of his laboratory and Dr. C. J. WADDINGTON for many helpful discussions. He is also indebted to the D.S.I.R. for a grant covering the experiment. Finally, the author thanks Mr. F. LIKELY for his help in scanning the plates.

(1) M. F. KAPLON, J. H. NOON and G. W. RACETTE: *Phys. Rev.*, **96**, 1408 (1954).

(2) C. J. WADDINGTON: *Nuovo Cimento*, **6**, 748 (1957).

(3) C. J. WADDINGTON: *Progress in Nuclear Physics*, vol. **8** (to be published).

(4) C. J. WADDINGTON: *Nuovo Cimento*, **3**, 930 (1956).

(5) P. S. FREIER, E. P. NEY and C. J. WADDINGTON: *Phys. Rev.*, **114**, 365 (1959).

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

Neutron-Proton Scattering and the Determination of the Pion-Nucleon Coupling Constant (*).

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(ricevuto il 22 Ottobre 1960)

This article reports a determination of f^2, the pion-nucleon coupling constant, from the neutron-proton differential charge-exchange cross section, using a method proposed by Chew [1]. For this purpose, the neutron-proton differential cross section at 710 MeV has been measured in hydrogen at nine angles within the range 180° to 160° (c.m. system) and in deuterium at two angles within the same range.

The method of the experiment was to focus the 740 MeV proton beam of the Berkeley synchrocyclotron upon a liquid deuterium target. The neutrons ejected from the deuterium at an angle of 7° 8′ relative to the incident proton beam passed through a sweeping magnet, were collimated, passed through another sweeping magnet, and impinged upon a liquid hydrogen target. A counter telescope consisting of four scintillators and a velocity-selecting Čerenkov counter [2] detected the elastic protons ejected from the hydrogen with an efficiency of about 60%, rejected all π-mesons and in conjunction with a γ-ray converter, rejected a large γ-ray component in the neutral beam with an efficiency of greater than 99%. The efficiency of the counter telescope was measured as a function of angle and velocity by performing a proton-proton scattering experiment in which both the scattered and recoil protons were detected. The proton beam incident on the deuterium target was monitored by a secondary emission chamber [3] which was calibrated, in a low intensity beam, against an argon-filled ionization chamber.

The method used to acquire information about the energy spectrum of the neutron beam was to study experimentally the velocity spectrum of the protons emerging from the deuterium at the same angle as the neutron beam. The effective energy spectrum of the neutron beam was deduced from these measurements and could be represented by a Gaussian distribution with a standard deviation of 8 MeV and a most probable energy of (710 ± 8) MeV [4].

(*) This work was done under the auspices of the U.S. Atomic Energy Commission.

(1) G. F. Chew: *Phys. Rev.*, **112**, 1380 (1958).

(2) C. Wiegand: *Čerenkov counters in high-energy physics*, UCRL-8148 (February 1958).

(3) G. W. Tautfest and H. R. Fechter: *Rev. Sci. Instr.*, **26**, 229 (1955).

(4) Measurements of the momentum spectrum of protons scattered from deuterium will soon be conducted at this laboratory with a system having a 1% momentum resolution.

The use of deuterium as the neutron source provided a direct method of measuring the absolute neutron intensity. Both targets were filled with deuterium and the counter telescope set at an angle of 7° 8′. By charge symmetry the scattering amplitudes are identical for the two successive scatterings (proton-neutron at the first target and neutron-proton at the second target). Therefore the ratio of the number of protons from the second target to the number of protons incident on the first target is proportional to the square of the neutron-proton cross section in deuterium at 7° 8′ (laboratory angle). Once this cross section was known it was possible to predict the absolute neutron flux in terms of the incident proton flux.

The results of the measurements are listed in Table I. The angular distribution was measured to an accuracy of 5%. The errors of the listed data include those due to the normalization and the uncertainty in the neutron transmission of the γ-ray converter which was used for angles less than 5°. The cross section in deuterium at 180° is 0.48 ± 0.08 of the cross section in hydrogen. This decrease is a manifestation of the Pauli principle. Neutron-proton scattering at 180° in deuterium leaves behind two protons in the final state and some of these states are forbidden by the exclusion principle ([5]).

TABLE I. – *Neutron-proton differential cross section in hydrogen and deuterium at* 710 MeV.

Cross section (10^{-27} cm²/sr) c.m. system	φ, c.m. system angle (deg.)
Hydrogen	
$6.15 \pm .54$	180.00
$5.04 \pm .51$	175.89
$4.39 \pm .39$	172.94
$3.97 \pm .38$	170.60
$3.84 \pm .31$	168.25
$3.12 \pm .25$	165.90
$3.35 \pm .28$	163.25
$2.71 \pm .24$	161.37
$2.65 \pm .23$	158.90
Deuterium	
$2.97 \pm .42$	180.00
$2.85 \pm .14$	163.25

CHEW ([1]) proposes that the neutron-proton differential cross section be multiplied by $x^2=(1+\cos\varphi+\mu^2/MT)^2$ and extrapolated to $x=0$, which is the position of the one-pion-exchange pole. The residue of the pole is given by

$$\alpha_0 = \frac{4(f^2)^2}{T^2(T/2M+1)}, \tag{1}$$

where T is the laboratory-system kinetic energy, φ is the c.m. scattering angle, μ is the charged pion rest mass and M is the nucleon mass. If the neutron-proton cross section ([7]) is multiplied by x^2 the result is

$$x^2 \frac{d\sigma}{d\omega} = y(x) = \frac{g^4}{4E^2} \cdot \left(x - \frac{\mu^2}{MT}\right)^2 + xA(x) + x^2 B(x), \tag{2}$$

where E is the total energy of a nucleon and A and B represent the contributions of intermediate states other than the one-pion-exchange.

Fig. 1 is a plot of $x^2(d\sigma/d\omega)$ versus x for the data obtained at 710 MeV. A least-square fit in polynomials in x was made to the data on the IBM 704 computer. Polynomials up to, and including fifth order were sought. The criteria for the selection of the best fit are contained in a paper by CZIFFRA ([6]).

([5]) I. YA. POMERANČUK: *Žurn. Eksp. Teor. Fiz.*, **21**, 1113 (1951).

([6]) P. CZIFFRA and M. MORAVCSIK: *A practical guide to the method of least squares*, UCRL-8523 (October 1958).

The values of q/M, where q is the sum of the squares of the residues divided by the errors, and M is the number of degrees of freedom, are listed in Table II. The minimum of q/M is a criterion for determining the best fit. The values of q/M are consistent with the expected values.

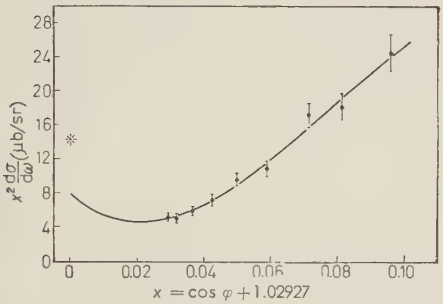

Fig. 1. – Plot of $x^2(d\sigma/d\omega)$ at 710 MeV versus $x = 1 + \cos\varphi + \mu^2/MT$. The end of the physical region (180°) is $x = 0.029\,27$. The solid curve is the cubic fit to the data. The asterisk denotes the residue corresponding to $f^2 = 0.08$.

Another test that can be applied to the polynomials in an effort to determine the best fit is the Fisher F test. The results of this test are listed in Table II

Table II. – *Results of least-square fits to* $x^2(d\sigma/d\omega)$ *at* 710 MeV.

Order of polynomial: n	q/M	$P(n)$	Coupling constant: f^2
1	1.447	0.01	imaginary
2	0.652	0.30	$0.032 \pm .012$
3	0.658	0.60	$0.059 \pm .022$
4	0.748	0.85	$0.096 \pm .050$
5	0.988	—	$0.128 \pm .169$

where $P(n)$ is to be interpreted as the probability of being correct if the polynomial of order n is selected to represent the data. The $P(n)$ reveal an unsatisfactory situation. If the data represent a low-order polynomial, then $P(n)$ should peak at a particular n. The indication is that the data do not represent a polynomial of order $n \leqslant 5$.

The contribution of multiple-meson exchange to the amplitude can be represented analytically by

$$A(x) \propto \int_0^\infty \frac{\alpha(x')\,dx'}{(x' + x + 3\mu^2/MT)} . \tag{3}$$

We assume that this branch cut can be approximated by a simple pole (as would be the case if there were a strong π-π resonance) located at an unknown value of x. We then define the function $u(x)$ by

$$u(x) = (x + a)^2 y(x) , \tag{4}$$

where a is the value of x at which the pole occurs. At $x=0$ the residue of $u(x)$ is a^2 times the residue of $y(x)$. We choose a value for a and make a least-square fit to $(x+a)^2 x^2(d\sigma/d\omega)$. The values of a used corresponded to ten values of the total energy of the multiple-meson intermediate state between 1 and 5 meson masses. The values of a corresponding to energies less than two meson masses have no physical

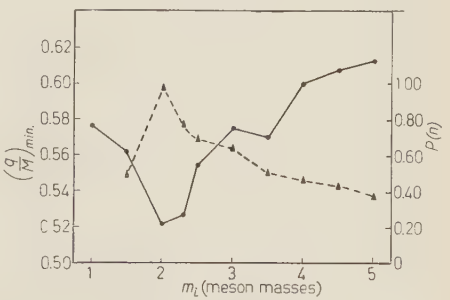

Fig. 2. – Statistical parameters of the best fit for a given m_i versus m_i: m_i is the total energy of the intermediate state in units of the charged meson mass. Circles are values of $(q/M)_{min}$; triangles are corresponding values of $P(n)$.

justification. For each case, the minimum of q/M was noted and the behavior of $P(n)$ studied. Fig. 2 shows the results

of this analysis. The second and third order polynomials in the neighborhood of $m_i = 2$ definitely exhibited the behavior

TABLE III. – *Results of least-square fit to* $(x+a)^2x^2\mathrm{d}\sigma/\mathrm{d}\pi$ *at* 710 MeV.

m_i (meson masses)	Order of best fit, n	$(q/M)_{\min}$	$P(n)$	f^2
1	3	0.576	0.64	—
1.5	2	0.562	0.55	$0.154 \pm .014$
2	2	0.522	0.97	$0.085 \pm .011$
2.3	2	0.527	0.76	$0.070 \pm .010$
2.5	2	0.556	0.69	$0.063 \pm .011$
3	2	0.575	0.64	$0.053 \pm .011$
3.5	2	0.571	0.50	$0.047 \pm .011$
4	2	0.600	0.46	$0.043 \pm .011$
4.5	2	0.616	0.43	$0.041 \pm .012$
5	2	0.626	0.38	$0.038 \pm .012$

one expects if the data represent a second order polynomial. The residue (and all other coefficients) showed a distinct plateau as a function of n. (This is to be compared with the results in Table II where no such plateau is present). The values of f^2 obtained for each m_i are shown in Table III. The errors on f^2 include the errors of the input data and a factor representing the goodness-of-fit.

The inclusion of the multiple-pion exchange by the preceding method definitely removes the ambiguity in the choice of the best fit. The value of f^2 in which one can have the most confidence, based on this analysis, is

$$f^2 = 0.085 \pm 0.011 .$$

CZIFFRA [7] has used Chew's proposal to analyze data at 90 MeV and 400 MeV and finds f^2 approx. 0.06 with about 10% error.

[7] P. CZIFFRA and M. MORAVCSIK: *Phys. Rev.*, **116**, 226 (1959).

IL NUOVO CIMENTO Vol. XVIII, N. 5 1° Dicembre 1960

Measurements of the Branching Ratios of K^0 and Λ^0.

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(ricevuto il 24 Ottobre 1960)

The branching ratios:

$$B_K = \frac{w(K_1^0 \to \pi^0 + \pi^0)}{w(K_1^0 \to \pi^0 + \pi^0) + w(K_1^0 \to \pi^+ + \pi^-)},$$

$$B_\Lambda = \frac{w(\Lambda^0 \to n + \pi^0)}{w(\Lambda^0 \to n + \pi^0) + w(\Lambda^0 \to p + \pi^-)},$$

have been measured by various groups, both indirectly [1,2] and by directly observing the γ's from the π^0 decays [1-3]. The B_K results varied widely and some of them seemed to contradict the $|\Delta I| = \frac{1}{2}$ rule.

We have measured B_K and B_Λ by direct observation of the γ's, in a $(34 \times 24 \times 20)$ cm bubble chamber containing a mixture of equal parts by volume of propane and methyl iodide. ($d = 1.1$ g/cm^3, radiation length 10 cm).

About 100000 photographs were taken in a 1.12 GeV/c negative pion beam from the « Saturne » synchrotron at Saclay.

The events analysed came from the reactions:

$$\pi^- + p \to \Lambda^0 + K^0 \quad \text{and} \quad \pi^- + p \to \Sigma^0 + K^0,$$

which were identified by using the decay angles, ranges of secondaries and V^0 coplanarity.

In the measurement of the branching ratio of the $K_1^0(\Lambda^0)$ only those events in which the associated $\Lambda^0(K_1^0)$ decayed via charged mode were used. The presence of at least 2 γ-rays was required.

The scanning efficiency was checked by scanning all the films twice.

Different corrections were applied to obtain the final values of the branching ratios [4]:

(*) On leave of absence from Istituto di Fisica dell'Università, Padova.

(1) F. Eisler, R. Plano, N. Samios, M. Schwartz and J. Steinberger: *Nuovo Cimento*, **5**, 1700 (1957).

(2) F. S. Crawford jr., M. Cresti, A. L. Douglass, M. L. Good, G. R. Kalbfleisch, M. L. Stevenson and H. K. Ticho: *Phys. Rev. Lett.*, **2**, 266 (1959).

(3) J. L. Brown, H. C. Bryant, R. A. Burnstein, D. A. Glaser, R. W. Hartung, J. A. Kadyk, D. Sinclair, G. H. Trilling, J. C. Vander Valde and J. D. van Putten: *Phys. Rev. Lett.*, **3**, 563 (1959).

(4) For the details about experimental set up and corrections, see: P. Musset: *Thesis* (University of Paris, 1960).

— spurious events due either to scattered tracks simulating a V^0 or γ-rays with two possible origins were corrected for;

— the observed number of charged mode V^0's was corrected for the fact that a V^0 with a short line of flight or a slow secondary, is missed;

— the most important correction, of course, is the one that comes from the variable probability of γ-ray observation. This probability was determined as follows.

We have taken the set of events in which both V^0 particles decayed through their charged modes. Each V^0 particle in turn was then assumed to have decayed through its neutral mode. From a Monte-Carlo calculation the direction and energy of the γ-rays produced were found; this combined with the theoretically deduced probability of γ-ray materialisation in our chamber gives the required probability of observation for each γ.

We obtained the following figures:

0.45 for a single γ,

0.63 for 2 γ's at least from a K^0,

0.19 for 2 γ's from a Λ^0.

The final results come from the observation of:

16 events of the type $\begin{cases} \Lambda^0 \to p + \pi^-, \\ K_1^0 \to \pi^0 + \pi^0 \\ \text{(at least 2 } \gamma\text{'s)}, \end{cases}$

and

7 events of the type $\begin{cases} K_1^0 \to \pi^+ + \pi^-, \\ \Lambda^0 \to n + \pi^0 \, (2\,\gamma) \end{cases}$

$$B_{K_1^2} = 0.26 \pm 0.06\,,$$
$$B_\Lambda = 0.28 \pm 0.08\,.$$

These results are compatible with those of Brown *et al.* in a similar experiment ($B_K = 0.25 \pm 0.05$; $B_\Lambda = 0.33 \pm 0.07$).

The errors quoted are standard deviations.

They do not disagree with the results deduced from the indirect measurement at Berkeley:

$$B_K = 0.32 \pm 0.04; \qquad B_\Lambda = 0.373 \pm 0.031\,.$$

Our value of B_K does not disagree with the lower limit (0.289) expected with the assumption of the $|\Delta \boldsymbol{I}| = \frac{1}{2}$ rule with the small admixture of $|\Delta \boldsymbol{I}| = \frac{3}{2}$ required to account for the decay $K^+ \to 2\pi$.

* * *

We would like to thank Professor L. Leprince-Ringuet for his constant encouragement, discussions and advice. Our thanks are due to Dr. C. A. d'Andlau who designed and built the measuring equipement, and to the I.B.M. France Company, which donated computer time and programme facilities. We are greatly indebted to the « Commissariat à l'Energie Atomique » and in particular to Mr. Maillet, Mr. Lévy-Mandel, Mr. Stickel and their colleagues of the « Saturne » team.

LIBRI RICEVUTI E RECENSIONI

E. G. Richardson – *Relaxation spectrometry*. North-Holland Publishing Comp., Amsterdam. In 8-vo, 140 pagine con numerosi grafici e figure.

Il volume consta di otto capitoli ed è dedicato allo studio dei fenomeni di rilassamento sia nel campo delle basse frequenze, come nel campo delle frequenze acustiche e ultrasonore. Si tratta quindi di un piccolo trattato di una branca dell'acustica, quella direi più viva oggi e più ricca di interesse per il fisico impegnato nello studio degli stati condensati della materia, o per il chimico fisico che si occupi di strutturistica. Solo il settimo capitolo esce dal campo dell'acustica, prendendo in considerazione brevemente il rilassamento dielettrico.

Ecco in dettaglio come viene presentato l'insieme degli argomenti trattati: nel primo capitolo l'Autore discute brevemente e con molta chiarezza la natura dei fenomeni di rilassamento in condizioni quasi-statiche, mostrando la relazione esistente fra moduli di elasticità e coefficiente di viscosità in corpi che non siano nè perfettamente elastici nè puramente viscosi. Nel secondo capitolo vengono illustrati semplici modelli meccanici od elettrici, costituiti rispettivamente da sistemi di smorzatori idraulici e molle, oppure da condensatori e resistenze elettriche capaci di un comportamento analogo a quello di un corpo dotato di elasticità e di viscosità allo stesso tempo, e se ne studia il comportamento; si esaminano inoltre semplici casi di fenomeni visco-elastici presentati da alcuni liquidi.

Nel terzo capitolo, dedicato alle basse frequenze, si introducono due grandezze che è comodo determinare sperimentalmente e cioè il decremento d'ampiezza delle vibrazioni libere, e la larghezza dell'intervallo di frequenze entro cui si ha il fenomeno di risonanza del sistema materiale sottoposto ad una sollecitazione periodica, e si mostra la relazione di queste grandezze con le proprietà visco-elastiche del materiale.

Nel quarto capitolo, dedicato alle frequenze acustiche, si studia dapprima lo smorzamento delle oscillazioni dei corpi solidi per effetto delle interazioni col mezzo circostante; indi si studia l'effetto delle cause interne di dispersione d'energia entro lo stesso materiale vibrante. Viene seguito il variare dei coefficienti elastici al variare della temperatura e della struttura cristallina del materiale, in particolare l'Autore si sofferma sul caso assai attuale da variazioni delle proprietà elastiche per effetto di danneggiamento prodotto da radiazione penetrante.

Il quinto capitolo è dedicato al campo di frequenze ultrasonoro. Vi è studiato l'assorbimento acustico nei gas e nei liquidi nonchè nei fluidi in prossimità del punto critico. Si esaminano alcuni meccanismi di decadimento dell'energia acustica a calore. L'esame dello spettro ultrasonoro di frequenze viene continuato nel capitolo sesto, dedicato ai

solidi. Si espongono le difficoltà che sorgono per l'accoppiarsi di onde longitudinali e trasversali e si studiano i casi dei metalli e degli alti polimeri ed il comportamento di liquidi e solidi nei pressi dei rispettivi cambiamenti di stato.

Il capitolo settimo, come già si è detto, è dedicato ad un brevissimo esame dei fenomeni di rilassamento dielettrico soprattutto dal punto di vista delle relazioni che esso presenta con il rilassamento acustico.

Il breve capitolo ottavo è più che altro un'appendice dedicata ad alcuni metodi di analisi dei risultati ottenuti negli esperimenti sui fenomeni di rilassamento.

Riassumendo, il volume di Richardson ci sembra cadere al momento giusto in un campo in cui si sentiva la necessità di un filo conduttore nel mare magnum dei molti e disparati risultati recenti.

F. Gaeta

GRIMSEHL – *Lehrbuch der Physik* - Band IV: *Struktur der Materie.* 12ª ediz. B. G. Teubner, Lipsia, 1959; prezzo 25,20 DM.

Il libro vuole essere una presentazione aggiornata e piana di alcuni capitoli della fisica moderna. Il maggior spazio è concesso al primo capitolo, relativo alla struttura della materia ed alla fisica nucleare, che introduce senz'altro alle proprietà statiche dei nuclei, per passare poi in rassegna nell'ordine i metodi di deflessione per particelle cariche, l'interazione delle particelle con la materia, i dispositivi sperimentali per la loro rivelazione e produzione, la radiazione cosmica, le disintegrazioni naturali ed infine le reazioni nucleari. Un breve ma chiaro capitolo introduttivo alla meccanica ondulatoria ed alla meccanica delle matrici, è seguito da un capitolo dedicato alla spettroscopia atomica. Nel capitolo sulla fisica molecolare vengono discusse le varie teorie sui legami, le proprietà delle molecole e delle macromolecole, gli spettri molecolari e gli spettri Raman. Seguono due capitoli in un certo senso complementari sulla fisica dei liquidi e dei solidi, con una presentazione prevalentemente classica delle proprietà e dei fenomeni.

Chiude il libro un modesto capitolo sulla costituzione della materia stellare che si conclude con un cenno alla radioastronomia, ramo moderno dell'astrofisica.

Come visione d'insieme la materia è esposta bene, con chiarezza e, data la sua vastità, in modo abbastanza completo ed equilibrato.

Il formalismo matematico è ridotto al minimo, e pertanto il libro è diretto agli studenti ed ai cultori della fisica moderna piuttosto che agli specialisti nei vari campi trattati.

E. Clementel

Progress in Nuclear Energy. Serie I, vol. 2: *Physics and Mathematics.* Editori: D. J. Hughes, J. E. Sanders e J. Horowitz. Pergamon Press, Londra (1959); prezzo 90 s.

Forse in nessun campo, quale quello della fisica nucleare applicata, è sentita l'esigenza di una periodica messa a punto degli sviluppi delle ricerche teoriche e sperimentali. In questo secondo volume della serie vengono anzitutto fornite preziose informazioni sulle sezioni d'urto dei neutroni con gli isotopi degli elementi pesanti, in particolare torio, uranio e plutonio, in due capitoli dovuti a ricercatri del Centro di Harwell (W. D. Allen), Los Alamos (R. L. Henkel), Oak Ridge (J. A. Harvey) e Brookhaven (R. B. Schwartz).

L'importante questione di come misurare lo spettro energetico dei neutroni nei reattori è discusso da M. J. Poole, M. S. Nelkin e R. S. Stone per la prima

volta da un punto di vista critico, soprattutto per quanto riguarda l'interpretazione delle misure sperimentali.

La situazione un po' confusa sulla standardizzazione delle sorgenti di neutroni viene finalmente chiarita da un articolo dovuto a R. Richmond, il quale fornisce un'utile tabella dei valori assoluti di alcune sorgenti. Nessuno meglio di G. von Dardel e N. G. Sjöstrand, che furono i pionieri della tecnica delle sorgenti pulsate, poteva inquadrare e discutere lo stato attuale delle ricerche con tali sorgenti, ricerche in corso ormai in diversi laboratori per lo studio delle proprietà dei mezzi moderanti. Un interessante confronto fra il lavoro sperimentale e teorico sviluppato negli Stati Uniti ed in Russia è fatto da J. B. Sampson e J. Chernick nel capitolo sul problema della probabilità di fuga alle risonanze da parte dei neutroni nei reattori termici. In realtà nell'articolo si parla di « Western approach » e di « Russian approach » ed è confortante una volta tanto, dato che « the truth lies somewhere between », l'accordo abbastanza buono dei risultati. I metodi per il calcolo dei reattori eterogenei, con particolare riguardo ai reattori cilindrici completamente riflessi, vengono discussi in un articolo dovuto ad A. Hassitt, il quale giustamente si preoccupa di valutare l'attendibilità delle approssimazioni che vanno introdotte nel calcolo dei reattori, soprattutto qualora non si disponga di una calcolatrice elettronica. Chi invece è così fortunato da possederla, troverà molto interessante l'ultimo capitolo di G. Goertzel e M. H. Kalos sull'applicazione del metodo di Montecarlo nell'ambito della teoria del trasporto dei neutroni, dove ad una rigorosa impostazione teorica dei problemi si accompagnano utili esempi applicativi.

Al pari del primo, anche questo secondo volume sarà per alcuni anni un libro praticamente indispensabile per chi si occupa di fisica dei reattori.

E. Clementel

Formation and Trapping of Free Radicals. Edited by A. M. Bass and H. P. Broida. Academic Press, New York and London, 1960, pp. xvi-522, prezzo $ 16.

I radicali liberi — « semplicemente definiti come frammenti molecolari che hanno normalmente una vita molto corta (dell'ordine dei millisecondi), che possiedono una reattività molto elevata e che sono generalmente caratterizzati dalla presenza di un elettrone non accoppiato » — sono stati per lungo tempo oggetto delle speculazioni del chimico che li ha largamente invocati per spiegare i dettagli cinetici ed i meccanismi delle sue reazioni. Questa raccolta di Brass e Broida — « leaders » di un gruppo che dal 1956 lavora presso il National Bureau of Standards sull'argomento del titolo — tratta di uno dei modi con i quali si è recentemente reso possibile lo studio diretto di queste specie chimicamente labili, quello di stabilizzarle mediante il loro incorporamento in matrici inerti e vischiose mantenute a temperature estremamente basse.

Una parte notevole del volume, che consta di 16 rassegne corredate da copiosa bibliografia, è appunto dedicata alla trattazione delle tecniche di intrappolamento ed alla descrizione delle delicate apparecchiature impiegate.

I metodi finora più usati per tentare di identificare le specie intrappolate e per avere delle informazioni sulla loro struttura, la spettroscopia ottica e l'assorbimento di risonanza paramagnetica, trovano inoltre posto in due tra i più riusciti capitoli del libro.

Altre rassegne trattano di problemi applicativi collegati con lo studio dei radicali stabilizzati come la loro importanza nell'astrofisica e nella biochimica, il loro possibile sfruttamento come sorgenti di energia per la propulsione a razzo, il loro ruolo nei processi conseguenti ai danni prodotti dalle radiazioni sulla materia.

La divisione del materiale fra i vari capitoli risente della natura ancora molto empirica delle ricerche in questo campo e può sembrare a volte alquanto arbitraria. Il volume risulta un poco frammentario ma raggiunge certamente lo scopo che si prefigge che è quello specialmente di stimolare nuove ricerche in questo campo non ancora ben classificato nell'ambito delle scienze chimiche e fisiche e che offre vaste ed attraenti possibilità.

Chi si interessa poi da tempo a questi problemi lo troverà pure molto utile specialmente per la completezza delle notizie bibliografiche la maggior parte delle quali si riferisce agli anni tra il 1956 ed il 1959.

G. Giacometti

K. Mendelsshon – *Cryophysics*. Interscience Pub. Inc., New York, 1960, pp. viii-183.

Il Dr. Mendelsshon ha cercato di condensare in questo libretto tutti gli aspetti della fisica delle basse temperature, dalla termometria al magnetismo, dai calori specifici alla superconduttività ed all'elio liquido. Egli è certo la persona che può scrivere un tale libretto perchè ha contribuito personalmente a tutti questi aspetti come pochi altri, ed ha vissuto lo sviluppo di questa parte della fisica con grande passione.

C'è tuttavia da chiedersi perchè si debba scrivere un così piccolo libro per una materia ormai tanto grande. Non certo per gli studenti, perchè mancano le derivazioni delle leggi più importanti ed i dettagli degli esperimenti più significativi, e neanche per i ricercatori perche mancano i riferimenti più recenti. Forse può essere utile a ricercatori di campi affini che vogliano farsi un'idea sommaria delle cose. Personalmente siamo contrari a queste idee così sommarie: ad esempio, a chi può servire leggere due pagine solamente sul film dell'elio, o mezza pagina sulle camere a bolle?

A parte questa critica generale, bisogna però dire che i diversi capitoli sono scritti con chiarezza, e l'ampiezza (o piccolezza) dedicata ad ogni argomento è realmente proporzionata alla sua importanza. Invece ci sembra che le citazioni siano fatte in modo partigiano, nel senso che un peso eccessivo viene dato al lavoro compiuto ad Oxford e che solo di questo lavoro vengono dati i più recenti dettagli.

G. Careri

Conference on Hyperconjugation. Indiana University, Bloomington, 2-4 June 1958. Conference Co-chairmen: V. J. Shiner and E. Campaigne. Pergamon Press, London, 1959, pp. viii-168 e varie figure, 35 sh.

Il volume raccoglie sedici comunicazioni fatte alla « Conferenza sulla iperconiugazione », tenutasi alla Indiana University (U.S.A.) nel Giugno 1958. L'aspetto più interessante che la sua lettura rivela è l'incontro di opinioni (o forse lo scontro) manifestatosi nella diversa risposta data alla domanda seguente: l'iperconiugazione è un dato di natura obbiettiva, oppure è la conseguenza di una particolare approssimazione descrittiva? Solo questo può giustificare la raccolta monografica delle comunicazioni e togliere in parte al lettore l'impressione di trovarsi di fronte al fascicolo di una rivista legato in volume.

Con il termine *iperconiugazione* si intende di solito designare la causa responsabile per gli effetti che si osservano sulla reagibilità, sugli spettri di assorbimento e su altre proprietà dei composti aromatici e non saturi quando un atomo di idrogeno del sistema viene sostituito da radicali alchilici (metile, etile, ecc.). L'iperconiugazione consiste

in una delocalizzazione degli elettroni dei legami C—H o C—C dei sostituenti verso il sistema aromatico o non saturo.

I due punti di vista estremi sono espressi nelle comunicazioni di Mulliken e di Dewar e Schmeising. Questi ultimi sostengono che l'iperconiugazione è la conseguenza di una descrizione schematica che ha trascurato di prendere in considerazione il diverso stato di valenza dei vari atomi di carbonio costituenti l'edificio molecolare. Una spiegazione dei fatti sperimentali può essere data valutando, anche su base semiempirica, la differenza di energia dipendente dal diverso stato di valenza degli atomi legati. Mulliken dal canto suo trova che la spiegazione precedente è la conseguenza di una eccessiva semplificazione, e procede ad una sistematica (e complessa) distinzione e classificazione degli effetti della delocalizzazione elettronica (considerata un dato obbiettivo) sullo stato fondamentale e sugli stati eccitati delle molecole.

Alcune comunicazioni sono rassegne su aspetti particolari del problema: evidenza termochimica degli effetti iperconiugativi (Turner); relazione tra iperconiugazione e lunghezza dei legami (Sutton); effetti della sostituzione isotopica di H con D (E. S. Levis; Shiner). In altri casi si tratta di tentativi di razionalizzazione, fatti dal punto di vista di una particolare approssimazione e con intenti relativamente limitati: trattazione semplificata dell'iperconiugazione secondo la teoria degli orbitali molecolari (Streitweiser e Nair); variazioni dell'energia di risonanza legate all'azione dei sostituenti e loro effetto sulla reagibilità (Taft e I. C. Lewis; Kreevoy); l'effetto dei diversi gruppi alchilici come conseguenza della iperconiugazione del legame C—H (Berliner); l'iperconiugazione nei legami N—H ed O—H (De la Mare). Vi sono infine comunicazioni di risultati sperimentali (solvolisi di alchil-alogeno derivati, spettri di assorbimento, effetti isotopici) la cui interpretazione riflette le idee degli autori sulla iperconiugazione.

Per chi è interessato all'argomento questa Conferenza segnerà senza dubbio una tappa importante. Va osservato tuttavia che il punto della situazione registrato nel testo non riflette un bilancio obbiettivo delle opinioni correnti, poichè quelli che condividono, in maggiore o minor misura, il punto di vista di Dewar e Schmeising erano assai scarsamente rappresentati tra i presentatori delle comunicazioni (probabilmente per lo scarso interesse al tema che è conseguenza di una tale posizione). Il testo inoltre non registra le discussioni vivaci di cui è detto nella prefazione.

Una maggior cura nella correzione delle bozze ed un indice sistematico più completo avrebbero aiutato considerevolmente il lettore, senza appesantire il costo, modesto, della pubblicazione.

L. Paoloni

F. W. Sears and M. W. Zemansky – *College Physics*. Addison-Wesley Publ. Co. Inc., U.S.A., 1959, pp. xxiii-1024; $ 9.75.

È questa la terza edizione di un libro ormai classico nell'insegnamento della fisica nei colleges americani. Esso è diviso in due parti presentate vuoi separatamente, vuoi unite in un unico volume.

La prima parte, di 448 pagine, riguarda la meccanica (14 capitoli), il calore (6 capitoli) e il suono (3 capitoli); la seconda parte, di 576 pagine, tratta l'elettricità e il magnetismo (16 capitoli), la luce (8 capitoli) e la fisica atomica (2 capitoli).

La trattazione, fatta senza uso del calcolo, presuppone la conoscenza soltanto dell'algebra e della trigonometria elementare. I sistemi di unità usati sono il sistema « inglese gravitazionale » (in cui le unità sono il piede per le lunghezze,

il secondo per i tempi e la libbra per le forze), il sistema CGS e il sistema MKS razionalizzato.

Il testo, sempre molto semplice e chiaro, è arricchito e completato da ben 760 belle figure e 3 tavole a colori. Alla fine di ciascun capitolo vi sono numerosi esercizi, ben scelti, le cui soluzioni sono date alla fine del libro.

Questo libro va considerato nel quadro del sistema scolastico americano il quale differisce molto notevolmente da quello europeo e, in particolare, da quello italiano. Secondo quanto affermano gli autori nella prefazione, gli argomenti trattati in questo libro sono abbastanza pochi da poter essere insegnati in due semestri. Invece, secondo le consuetudini prevalenti in Italia, e probabilmente anche in altri Paesi europei, gli argomenti trattati sembrano molti, tanto che appare non facile far apprendere agli studenti — in un solo anno o anche in due anni di studio — una così estesa e varia serie di nozioni, anche se presentate in forma elementare. Il libro in esame è più vasto dei più ampi testi in uso nei nostri Licei Scientifici con i quali, peraltro, è paragonabile per ciò che riguarda il tono della trattazione.

Un esempio tipico della impostazione generale del libro è dato dalla trattazione del secondo principio della termodinamica, la quale inizia con una efficace descrizione qualitativa delle trasformazioni, nei processi ciclici, di calore in lavoro e viceversa, descrizione che si conclude con la definizione di efficienza di una macchina che lavora fra due sorgenti a ben determinate temperature. Segue una discussione dei cicli che rappresentano, schematicamente, i motori a combustione interna, il motore diesel e la macchina a vapore. Viene quindi dato l'enunciato del 2° principio della termodinamica secondo Lord Kelvin e la descrizione della macchina di Carnot, presentata come quella macchina che, secondo quanto si può dimostrare, ha il massimo rendimento possibile fra tutte le macchine che lavorano fra le stesse due temperature; il carattere reversibile della macchina è accennato solo incidentalmente. L'enunciato del teorema di Carnot è quindi dato riferendosi alla macchina di Carnot come macchina ideale. Il capitolo termina con la definizione di scala assoluta delle temperature e con una discussione dello zero assoluto.

Gli autori cercano, ogni qual volta ciò sia possibile, di associare le leggi della fisica classica con le più moderne applicazioni. Un esempio tipico è costituito dal capitolo sul magnetismo. Esso inizia con la definizione del vettore $\boldsymbol{B}$ fatta a mezzo della forza agente su di una carica elettrica in moto; ad essa segue la definizione elementare (ossia senza far uso di integrali) del flusso di $\boldsymbol{B}$ e la discussione del carattere circolare delle orbite descritte da un corpuscolo carico che si muove in un campo magnetico uniforme perpendicolare alla velocità iniziale. Viene quindi data la descrizione del ciclotrone, dei metodi di misura del rapporto e/m dello spettrografo di massa. Si passa, quindi, alle forze che un campo magnetico esercita su circuiti percorsi da correnti.

Un altro esempio si ha nel capitolo sulle forze elettromotrici indotte ove vien data la legge di Faraday, prima per il conduttore in moto in un campo magnetico (caso del flusso tagliato) e quindi per il caso del « flusso variato ». Segue una descrizione relativamente dettagliata del betatrone senza, peraltro, praticamente accennare alla natura non conservativa del campo elettrico generato dalla variazione di flusso del vettore $\boldsymbol{B}$. Quindi viene stabilita la forza elettromotrice agente in una spira rotante e si accenna ai generatori di corrente alternata e continua, alla misura del campo magnetico e alle correnti di Foucault.

L'ultimo capitolo dell'elettricità, intitolato « elettronica », riguarda: le pompe a vuoto, l'emissione termoionica, la rettificazione, i triodi, i tubi a relay, i principi elementari dell'amplificazione e della

oscillazione, la modulazione in ampiezza, la detezione, le radio riceventi, i tubi a raggi catodici, l'effetto fotoelettrico, i tubi a raggi X e la conduzione nei gas.

Dei due capitoli di fisica atomica, l'uno riguarda essenzialmente la descrizione dell'atomo di Bohr e gli elementi della spettroscopia, l'altro gli elementi della radioattività, delle reazioni nucleari, dei raggi cosmici, della fissione e delle reazioni termonucleari.

In complesso questo libro è molto pregevole perchè, malgrado il carattere elementare della trattazione, riesce a fornire una notevole vastità di informazioni sui più diversi fenomeni fisici e sulle loro applicazioni sia classiche che moderne. L'omissione di alcune precisazioni, che, a mio giudizio, sembrano essenziali per una effettiva comprensione delle leggi fisiche, è da imputarsi, almeno in gran parte, alla elementarietà dei mezzi matematici usati.

Il testo, ottimo tipograficamente, è illustrato in maniera ricca ed efficace.

G. AMALDI

G. KENDALL WHITE – *Experimental Techniques in Low-Temperature Physics.* Clarendon Press, Oxford, 1959, pp. VIII-328, figg.128; 45 s.

L'uso delle basse temperature, e particolarmente di quelle che possono essere ottenute mediante raffreddamento con elio liquido, si va sempre più diffondendo nella ricerca scientifica e perciò un sempre maggior numero di sperimentatori si trova a dover risolvere i problemi pratici connessi con il loro impiego. Le tecniche da usarsi sono alquanto diverse da quelle usuali, quelle, per intenderci, che vanno bene per l'azoto liquido; l'elio liquido infatti è delicato e capriccioso, a causa, specialmente, del suo calore di evaporazione assai basso, e l'idrogeno, migliore sotto certi aspetti, è più pericoloso e perciò più difficile da trattare. Le proprietà dei materiali, poi, si modificano considerevolmente a temperatura molto bassa e non sempre ci si può fidare di certi vecchi amici. Qualunque esperto del ramo può confermare che per progettare un buon criostato o anche semplicemente per ottenere con successo il trasferimento da un recipiente all'altro, ci vuole non poca esperienza e tutti ricorderanno qualche caso in cui, per misteriose ragioni, le cose sono andate diversamente da quanto ci si poteva aspettare.

Il volume di G. KENDALL WHITE, pubblicato nella pregevole raccolta della Oxford Press, raccoglie molti dati pratici, elementi di progetto e informazioni generali, che possono essere di grande aiuto per chi si cimenti in questo tipo di esperienze, specialmente quando, come accade in Italia, l'elio liquido sia assai costoso e difficile da ottenere. È certo che, in qualche caso, si potrà far meglio, anche perchè qui non c'è tutto, ed il progresso, naturalmente, cammina, ma una accurata lettura di queste pagine potrà, altrettanto certamente, risparmiare molti errori il cui prezzo è già stato pagato da altri.

Il libro non ha pretese scientifiche pure e si limita a brevi, ma opportuni, richiami ai grandi problemi di fondo, esso tocca però tutti gli aspetti pratici in modo diretto e preciso, diventando così veramente un buon consigliere. La materia è divisa in tre parti: la prima tratta della produzione di basse temperature (liquefattori di gas), conservazione e travaso dei gas liquefatti, scambiatori di calore e misure di temperatura; la seconda è tutta dedicata ai criostati da laboratorio: essi differiscono non poco tra loro, a seconda delle esigenze della ricerca che possono imporre l'esistenza di finestre trasparenti a certe radiazioni, l'accurata regolazione della temperatura, spesso in un campo assai vasto, la trasmissione, all'interno della zona fredda,

di correnti elettriche o sforzi meccanici, ecc. La tecnica del vuoto, naturalmente, è un complemento essenziale di quella delle basse temperature e richiede, in questo campo, qualche speciale accorgimento, che viene esposto nel capitolo nono. La terza parte, più breve, è riservata a formule e tabelle sulle più interessanti proprietà dei materiali: capacità termica, coefficiente di dilatazione, conduttività elettrica e termica. I numerosi riferimenti bibliografici permettono di completare l'informazione fino al 1957, data, purtroppo, non recentissima.

Da questo si comprende che il libro non sarà in generale capace di rispondere a tutti i quesiti del ricercatore, ma che potrà tuttavia costituire una buona introduzione nel campo dei grandi freddi.

F. A. Levi

Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

Questo Fascicolo è stato licenziato dai torchi il 7-XII-1960